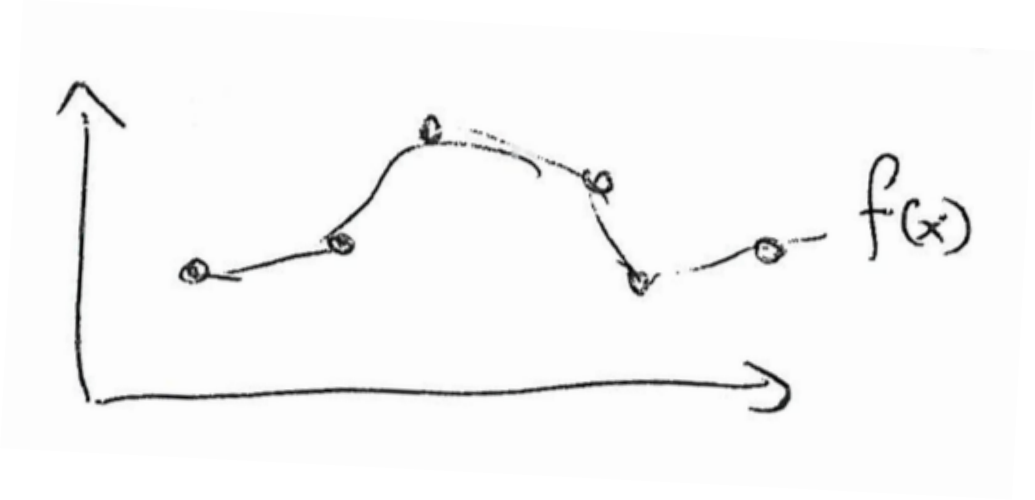
Numerical Methods II: interpolation

The basic problem is well known: given the values ($f_1,f_2,\ldots,f_N$) of a function $f=f(x)$ at the points ($x_1,x_2,\ldots,x_N$), where $f_i=f(x_i)$, find:

1) $f(\bar{a})$, where $\bar{a}$ inside $[x_1,x_N]$: **interpolation**
2) $f(\bar{a})$, where $\bar{a}$ outside $[x_1,x_N]$: **extrapolation**

Both interpolation and extrapolation must model a function among or beyond the assigned set of points. For this we need model functions that are sufficiently general to accommodate (e.g., to approximate) a large class of functions.



Exemples:

- polynomials
- rational functions
- trigonometric functions

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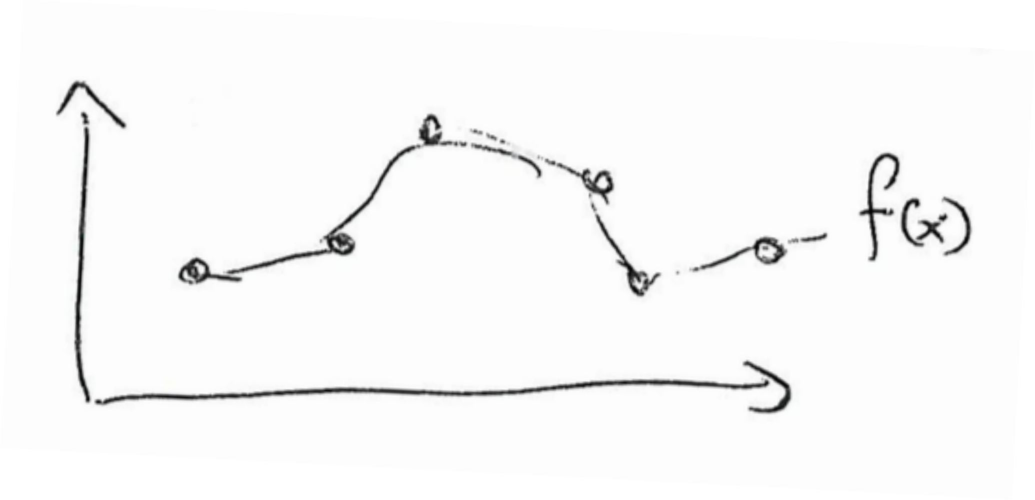

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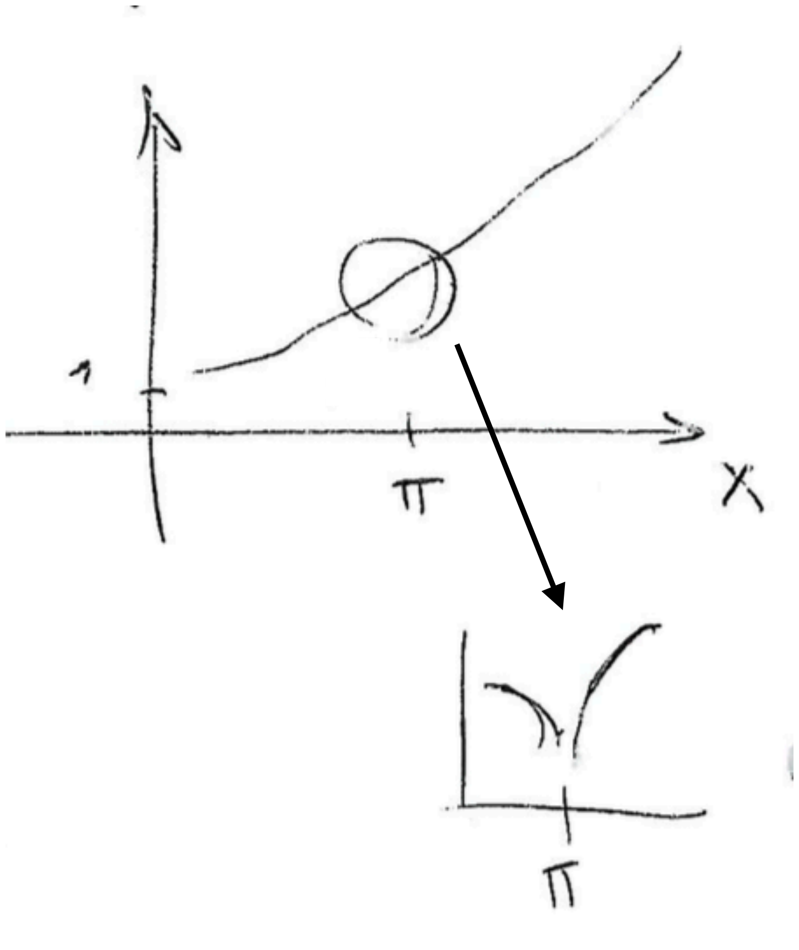
$$f(x) = 3x^2 + \frac{1}{\pi^4}\ln\left[(\pi - x)^2\right] + 1$$



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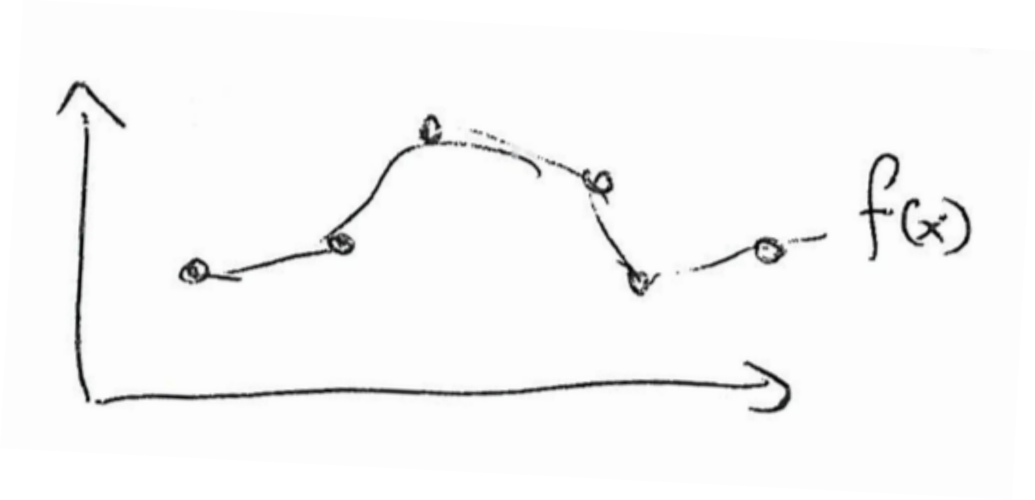

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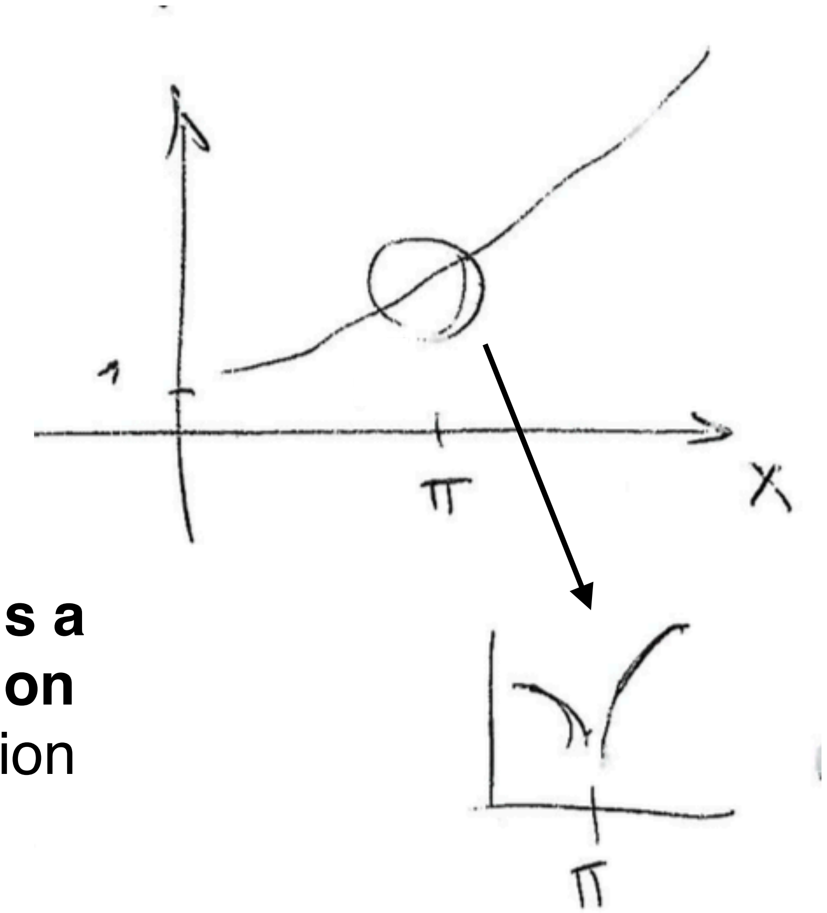
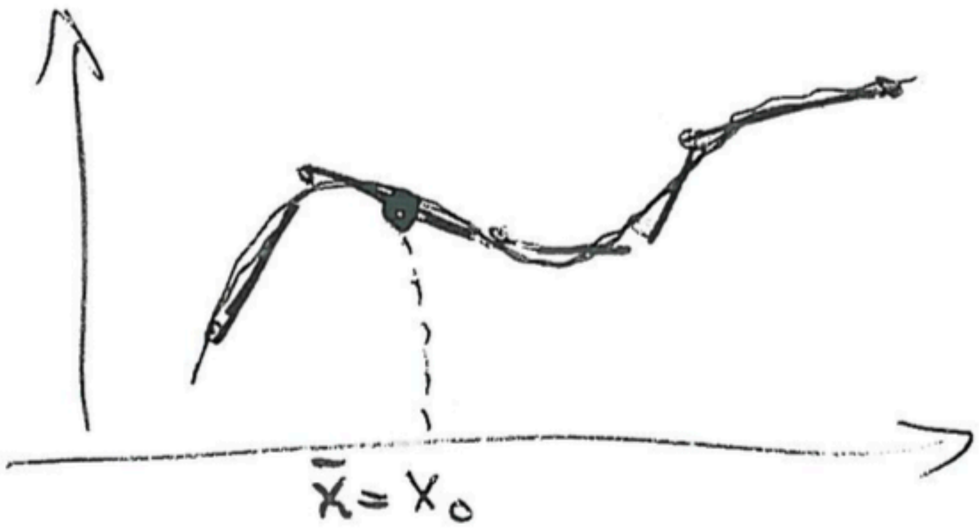


In other words, **numerical interpolation and extrapolation is a well-posed mathematical problem if the underlying function is smooth**. If this is not the case, extrapolation and interpolation are not reliable.

Theoretically, there are two steps:

1) find an interpolating function at the assigned points
2) evaluate this function at the desired point x_0



In practice, it is preferable to combine step 1) and 2):
$f(x_0)$ is evaluate directly from $(f_1,f_2,\ldots,f_N)$ and $(x_1,x_2,\ldots,x_N)$.
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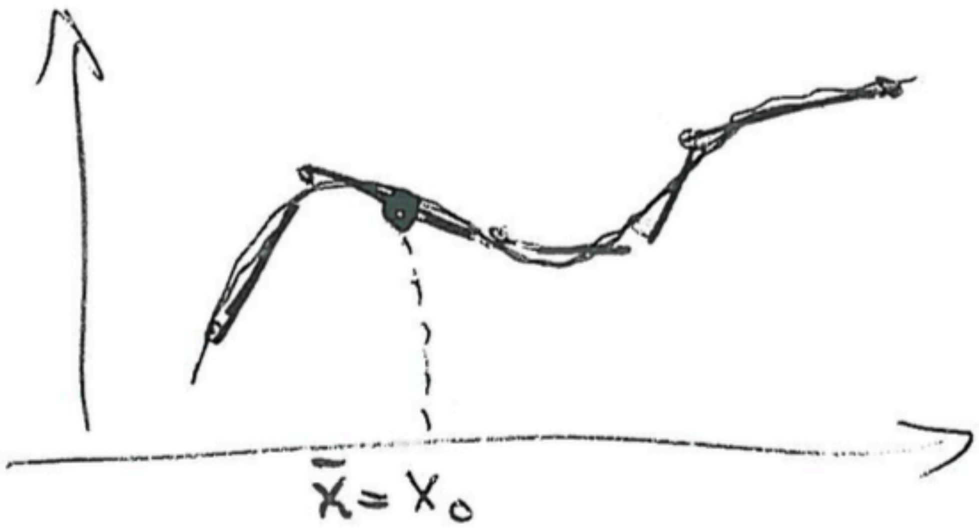

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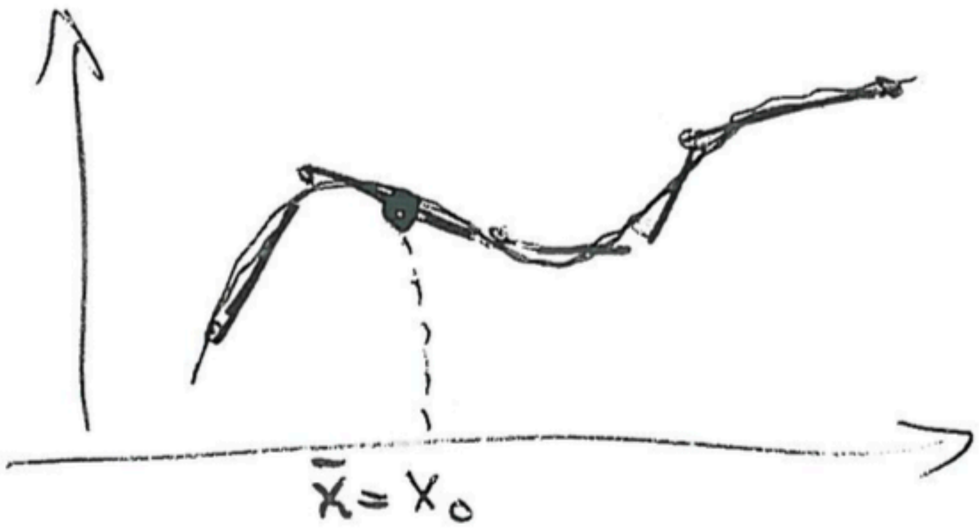

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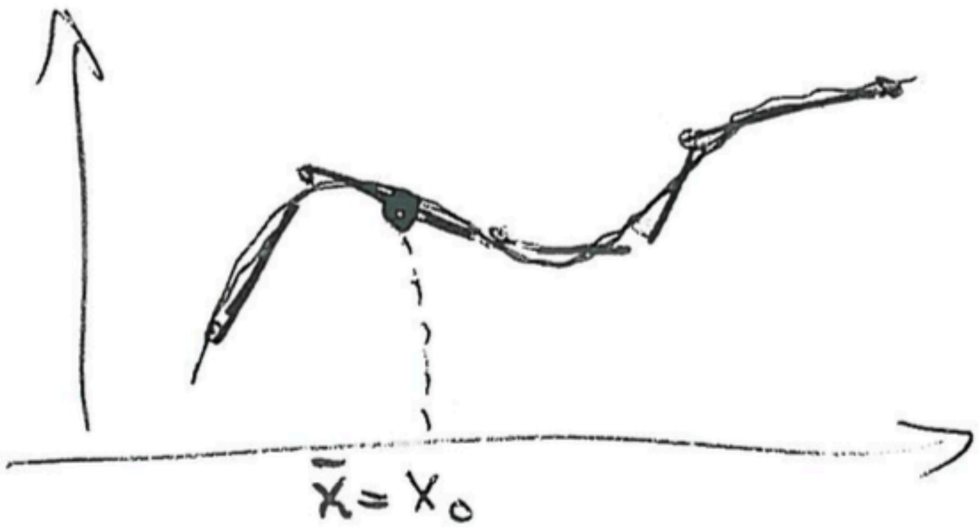

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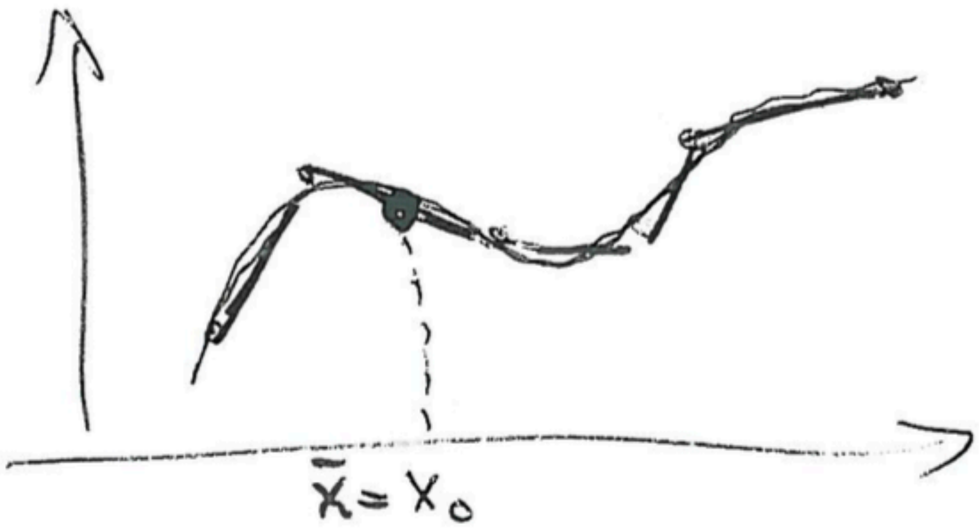

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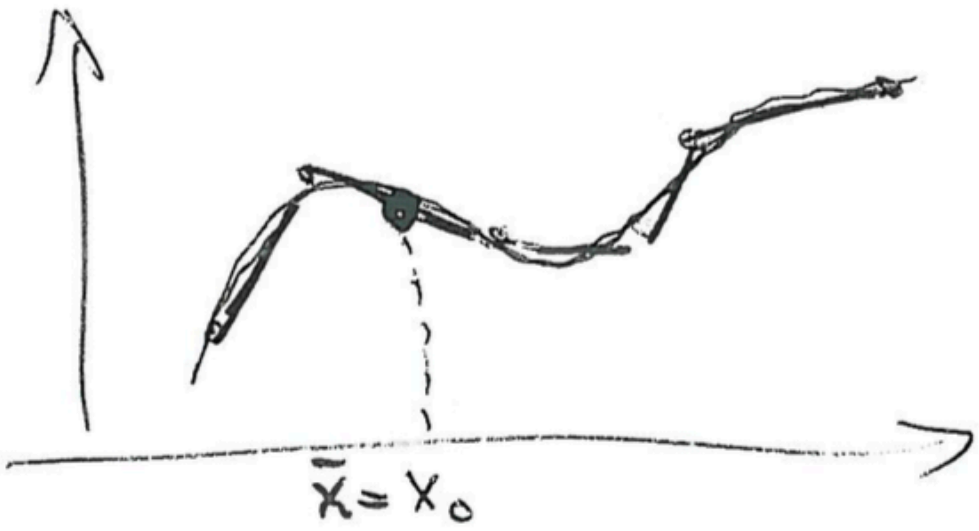

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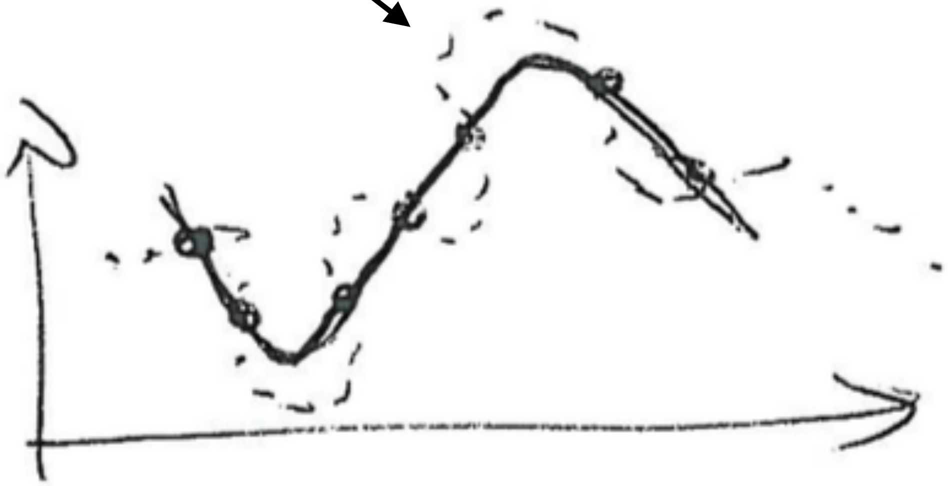
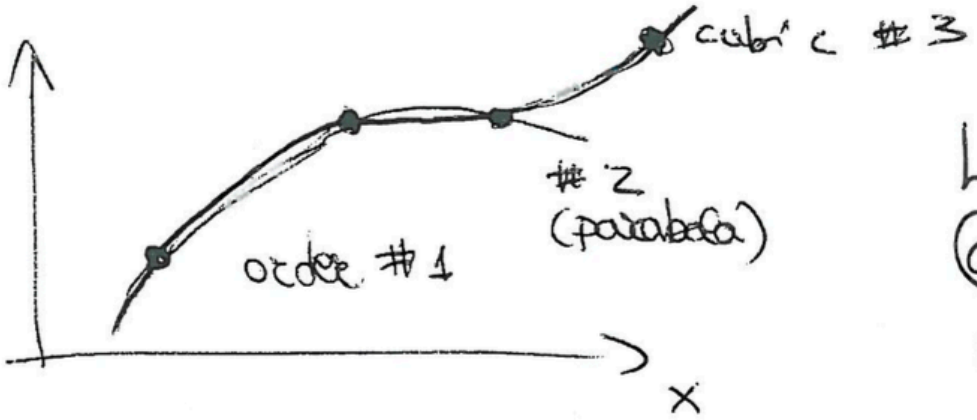
Ex. i) function with sharp corners (i.e., large gradients)
—> low order polynomial is a good idea

Ex. ii) function that is smooth —> use high order polynomial


with a high polynomial

For high order polynomials, never go past 5th order. If I have N points at which the function is tabulated, then the maximum order of the polynomial I can use is $N_{max} = N-1$



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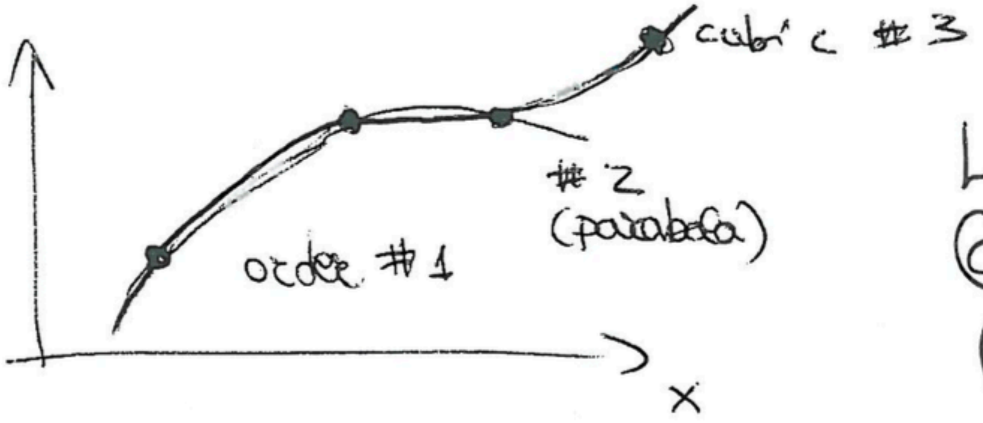

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Lagrange's Formula:

Given $(x_1,x_2,\ldots,x_N)$; $(f_1,f_2,\ldots,f_N) = (y_1,y_2,\ldots,y_N)$

$$y(x) = \frac{(x-x_2)(x-x_3)...(x-x_N)}{(x_1-x_2)(x_1-x_3)...(x_1-x_N)}y_1 + ... + \frac{(x-x_1)(x-x_2)...(x-x_{N-1})}{(x_N-x_1)(x_N-x_2)...(x_N-x_{N-1})}y_N$$

point at which I want to interpolate

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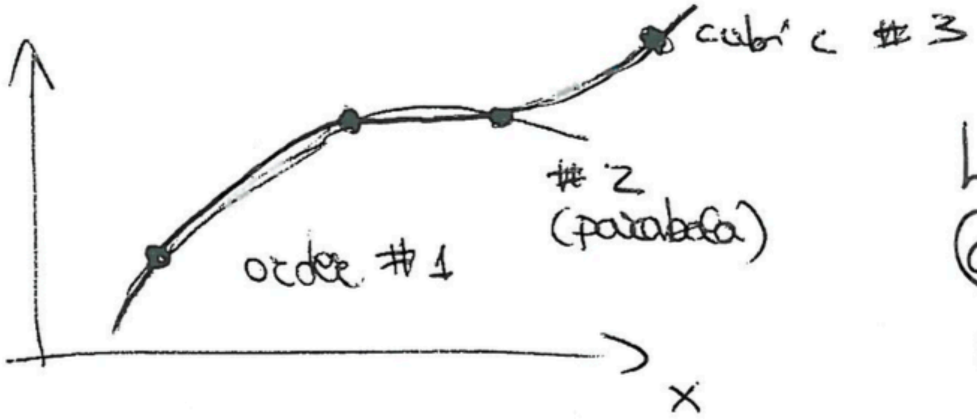

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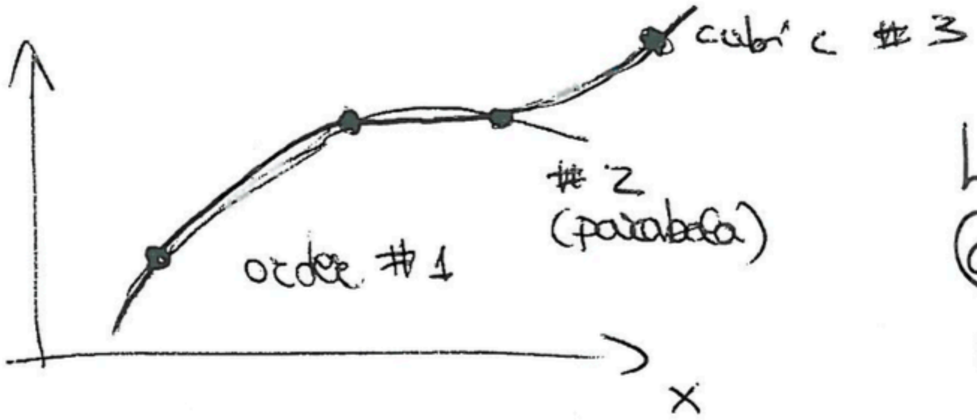

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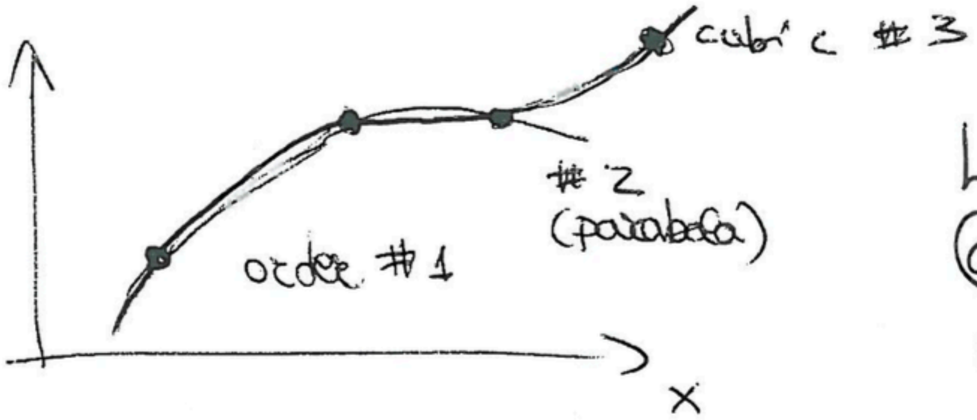

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EXAMPLE: N=2 (straight line)

$$y(x) = \frac{(x-x_2)}{(x_1-x_2)}y_1 - \frac{(x-x_1)}{(x_1-x_2)}y_2 = ty_1 + uy_2$$

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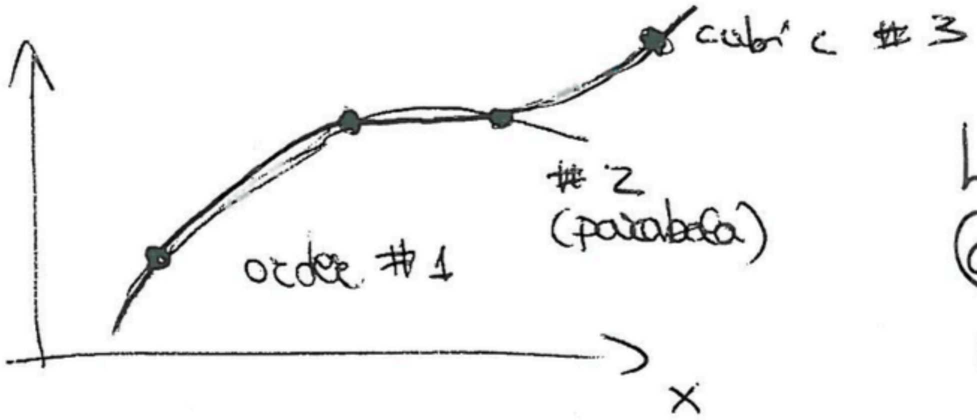

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$$(x_1,y_1),(x_2,y_2)$$

$$y = \alpha x + \beta$$

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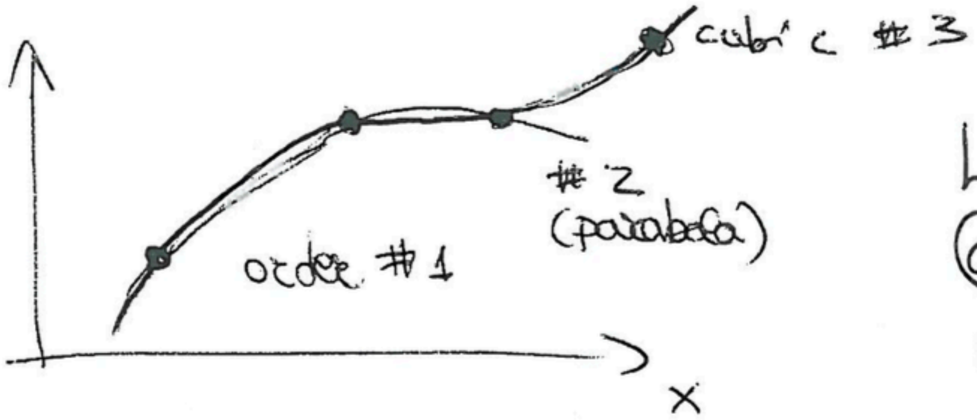

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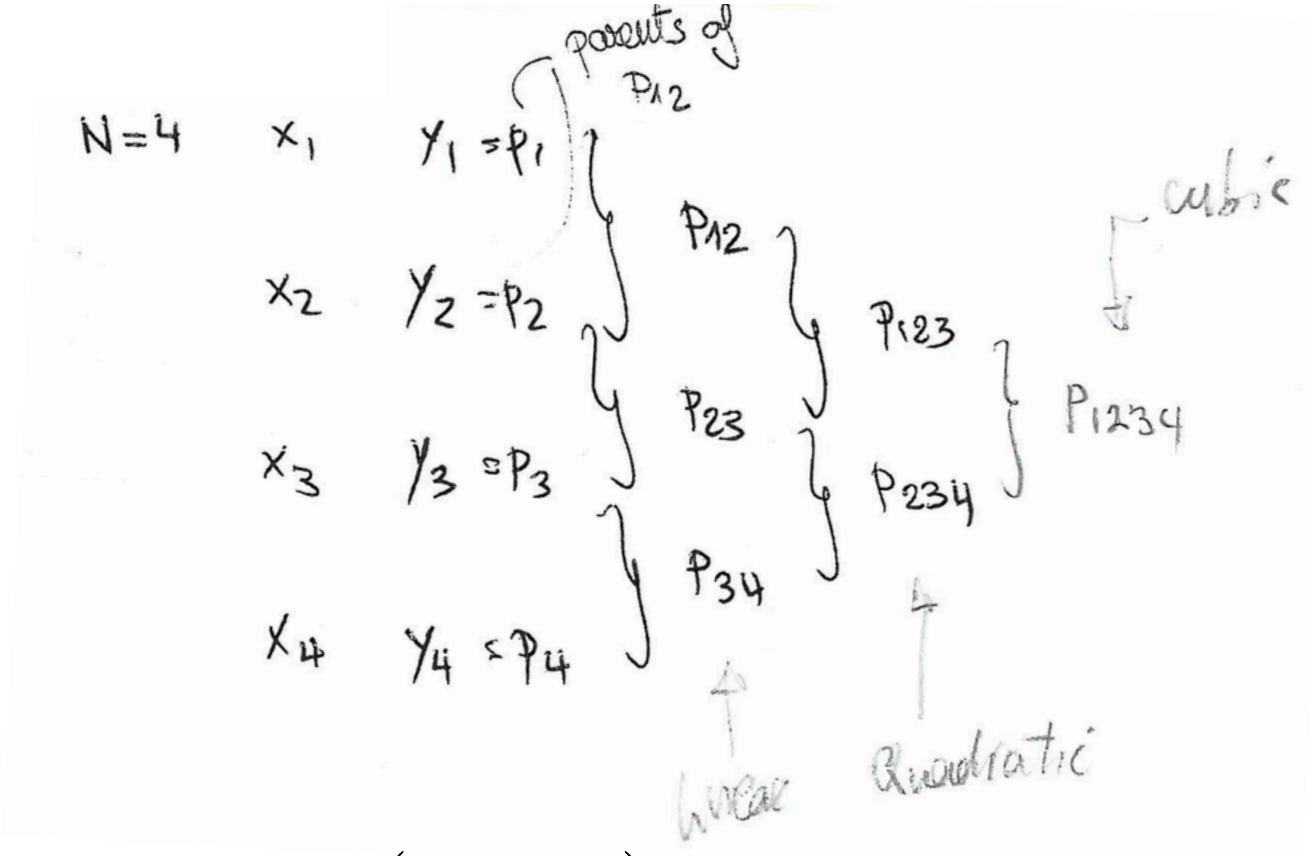
Lagrange's formula is fine mathematically, but it is not easy to implement numerically

Neville's Algorithm:

a smart implementation of Lagrange's formula



$$p_{i(i+1)\ldots(1+m)}(x) = \frac{(x - x_{i+m})p_{i(i+1)\ldots(i+m-1)} + (x_i - x)p_{(i+1)(i+2)\ldots(i+m)}}{x_i - x_{i+m}}$$

where m=N-1

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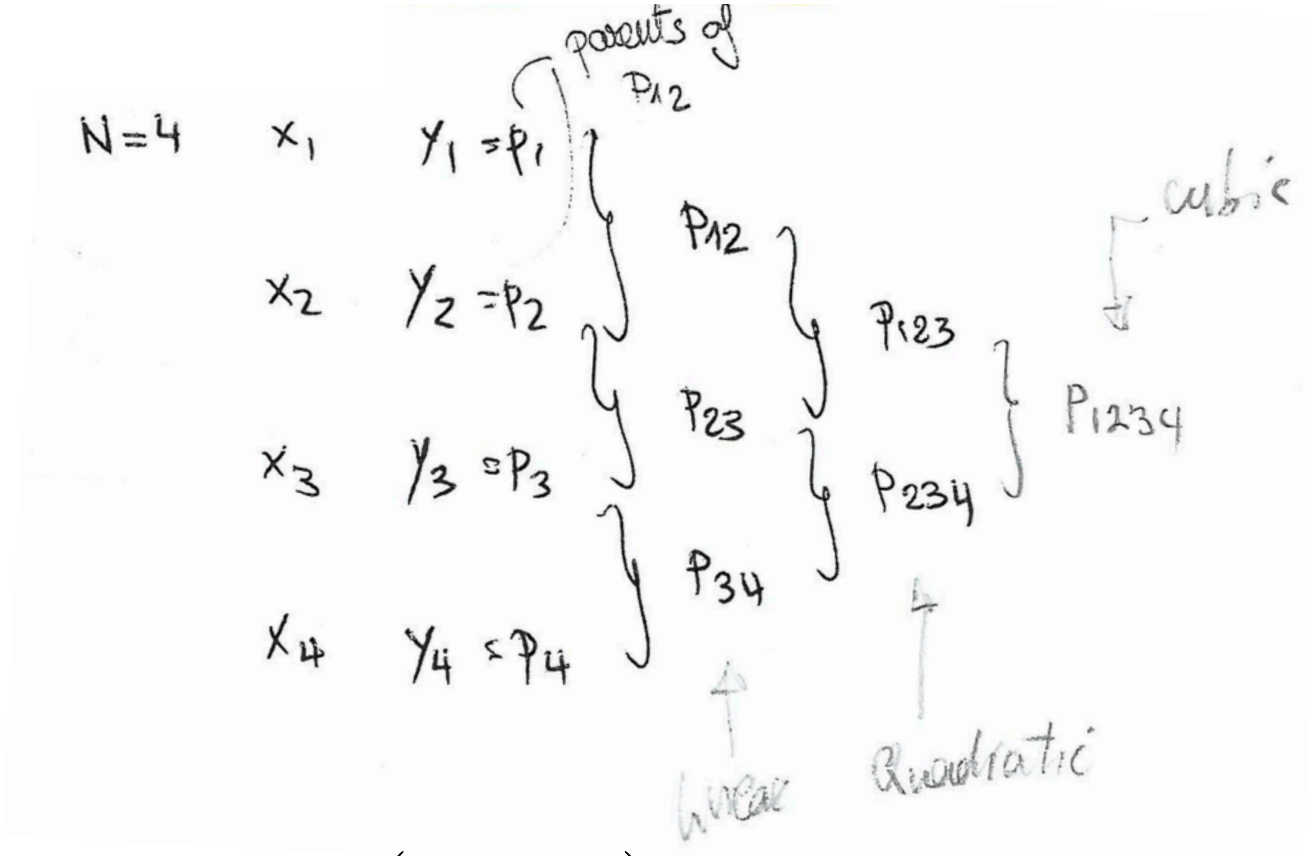

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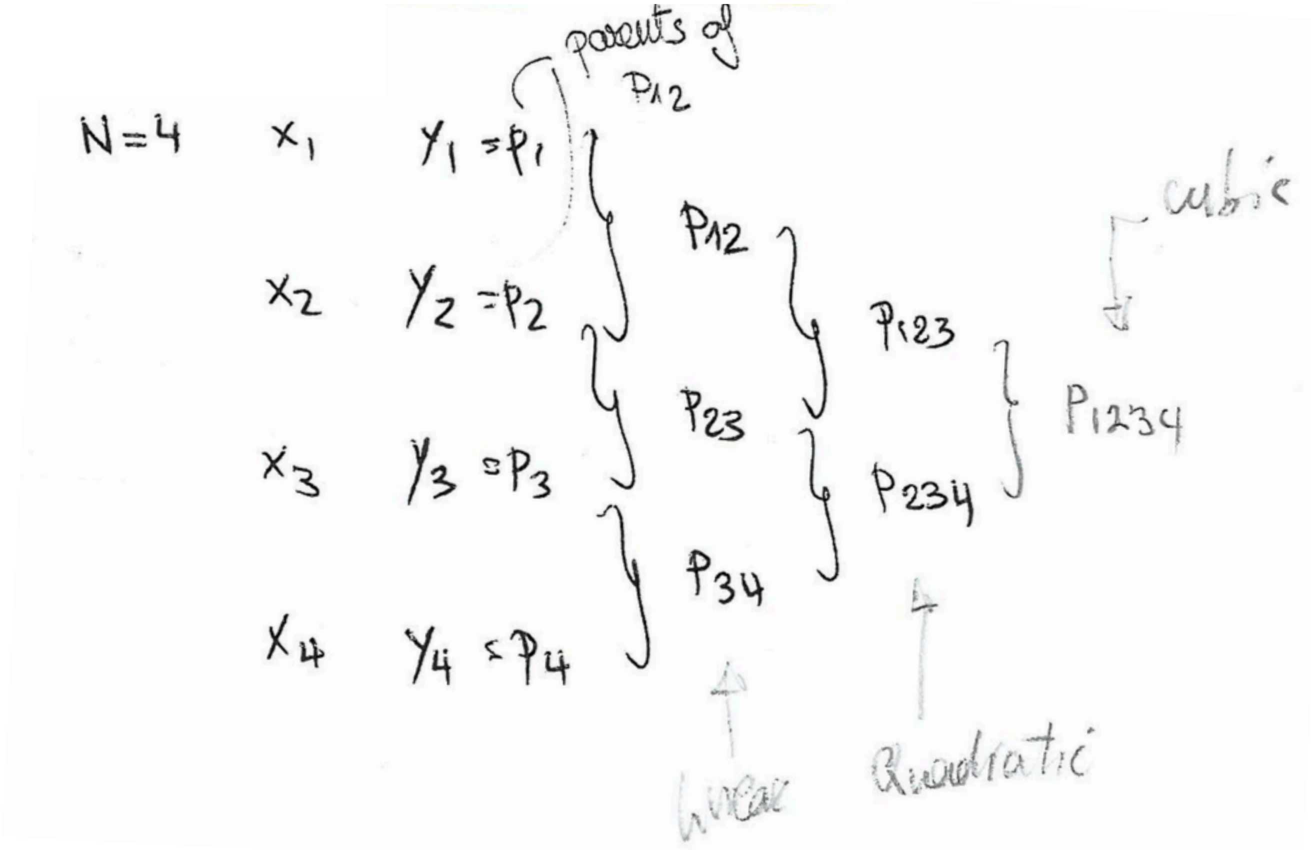

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$$p_{1234}(x) = \frac{(x-x_4)p_{123} + (x_1 - x)p_{234}}{x_1 - x_4}$$ cubic

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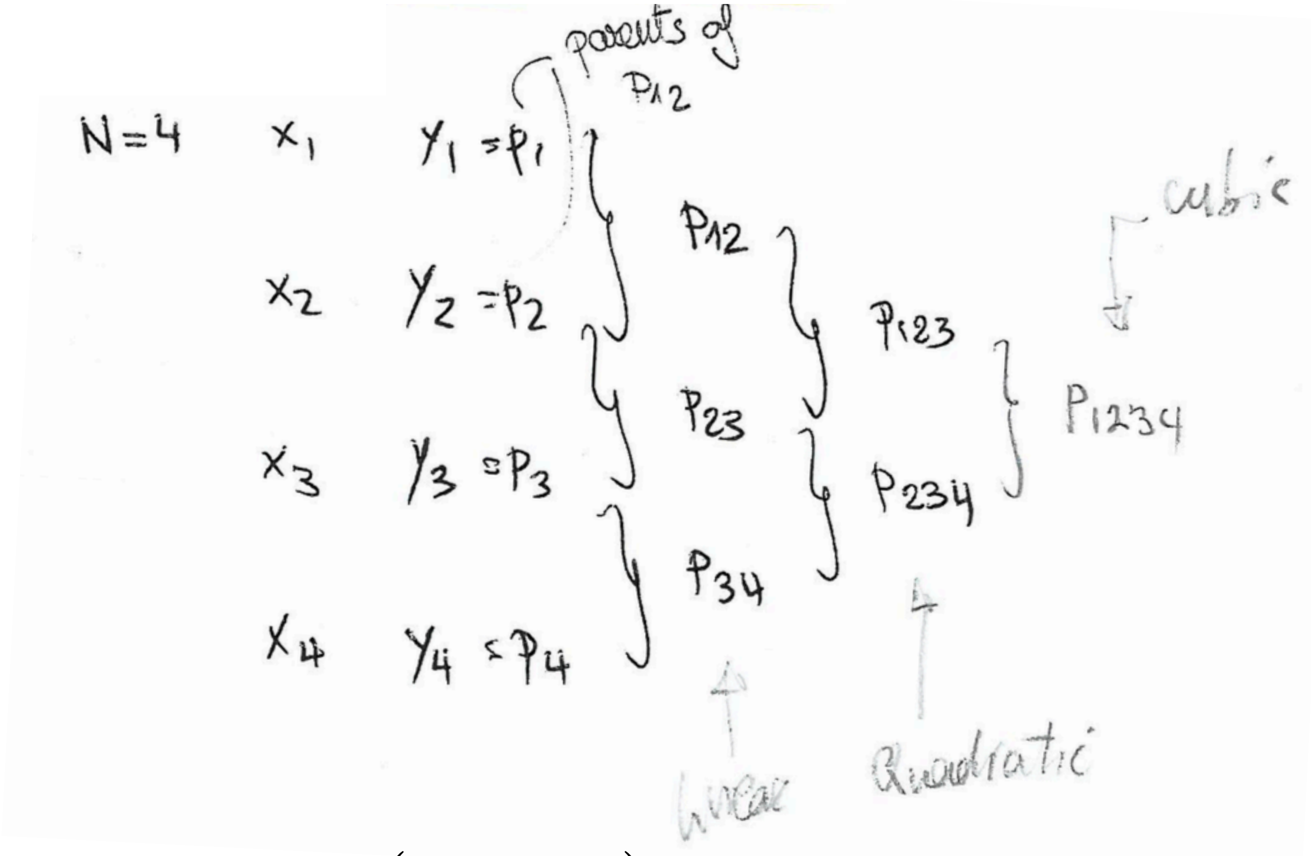

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cubic

$$p_{123}(x) = \frac{(x - x_3)p_{12} + (x_1 - x)p_{23}}{x_1 - x_3}$$

$$p_{234}(x) = \frac{(x - x_4)p_{23} + (x_2 - x)p_{34}}{x_2 - x_4}$$

parabola

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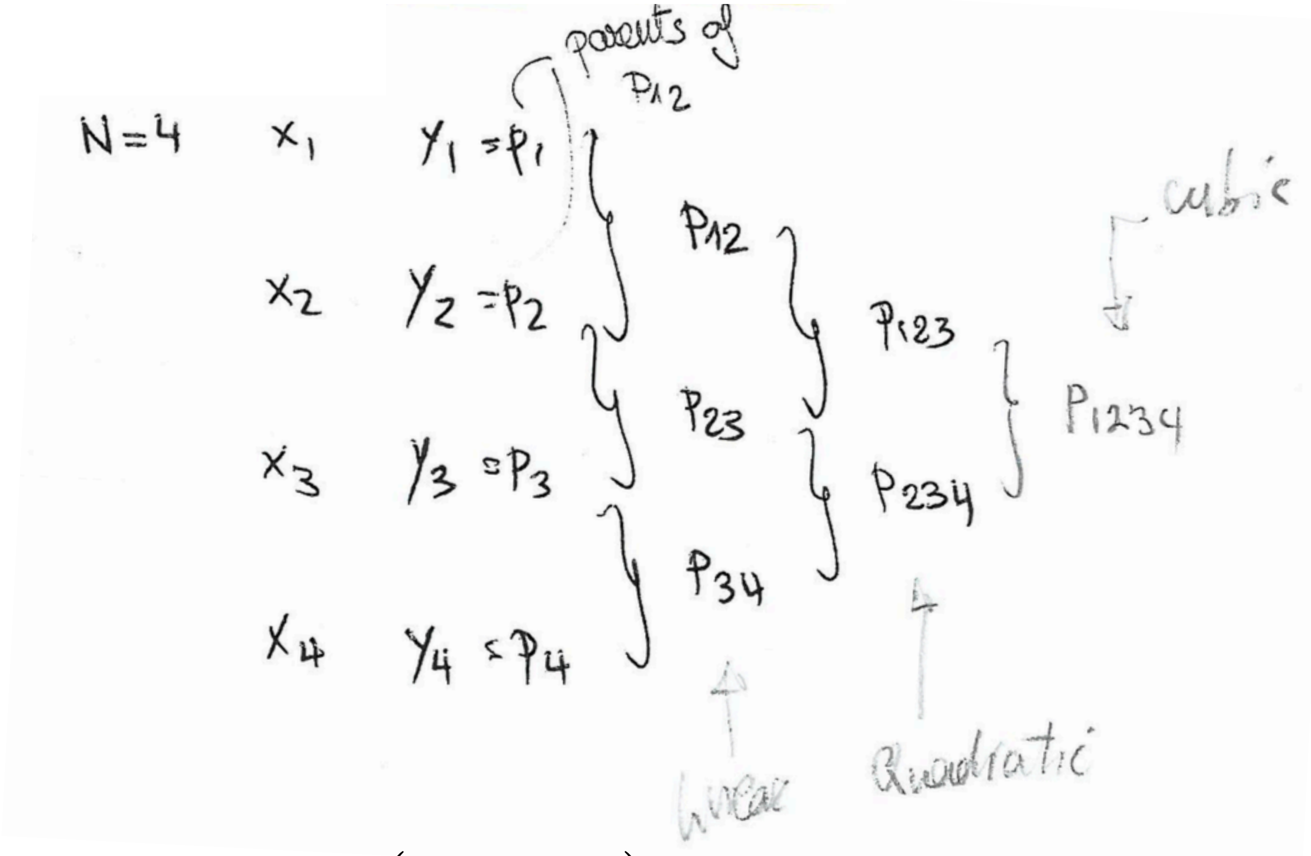

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parabola

$$p_{12}(x) = \frac{(x - x_2)p_1 + (x_1 - x)p_2}{x_1 - x_2} \qquad p_{34}(x) = \frac{(x - x_4)p_3 + (x_3 - x)p_4}{x_3 - x_4}$$

straight line

$$p_{23}(x) = \frac{(x - x_3)p_2 + (x_2 - x)p_3}{x_2 - x_3}$$

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Global interpolation methods: cubic spline [NOTE: you can call this function/routine when coding]

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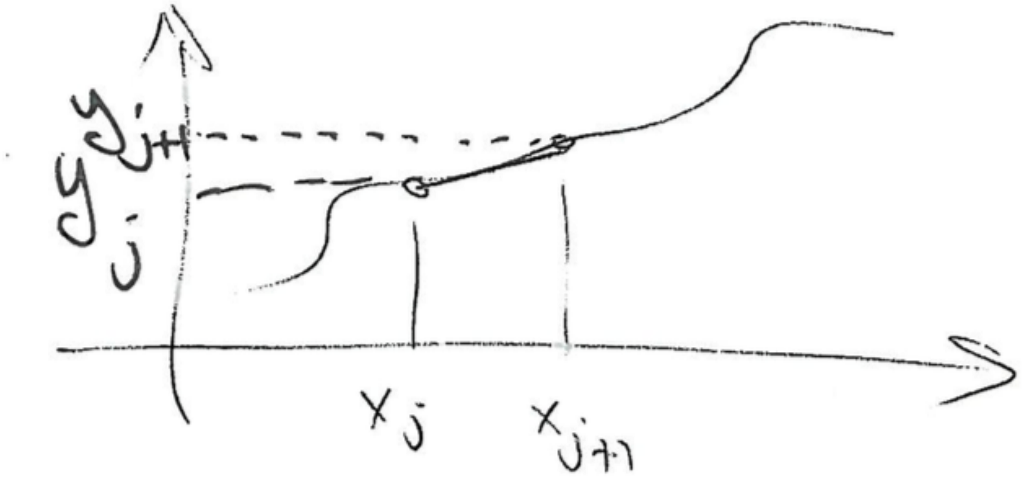
Global interpolation methods: cubic spline [NOTE: you can call this function/routine when coding]

Given (x_1,x_2,…,x_N) and (f_1,f_2,…,f_N) = (y_1,y_2,…,y_N), consider two points x_j,x_{j+1} and write a linear interpolation formula for there two points (this is a special case of the general Lagrange interpolation formula):



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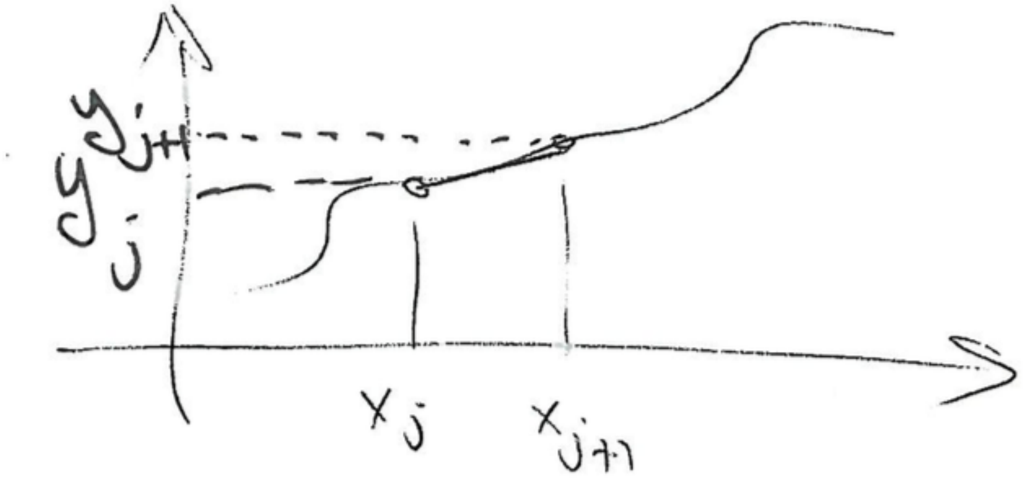

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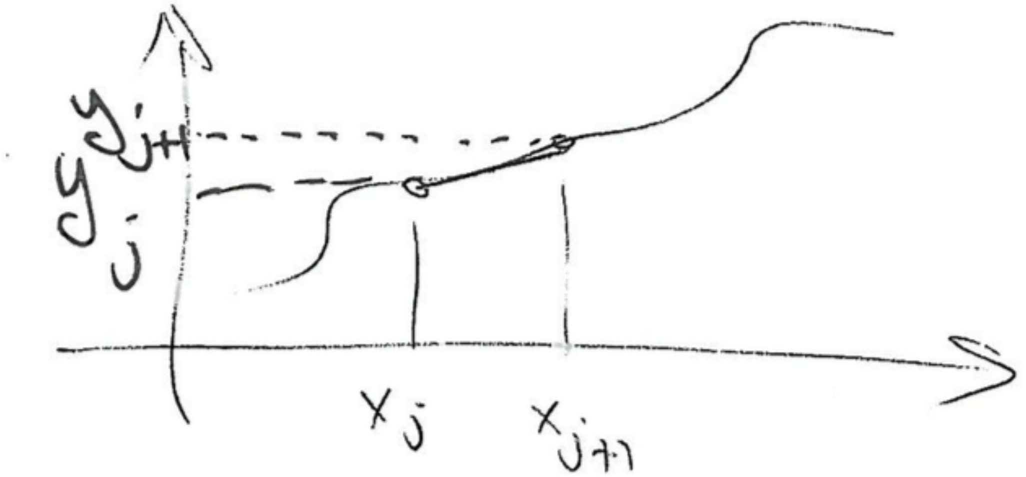

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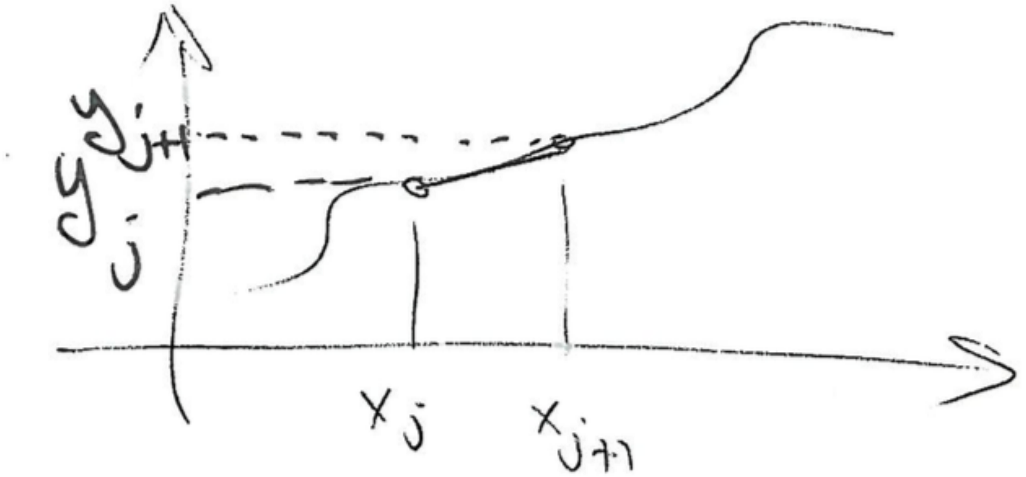

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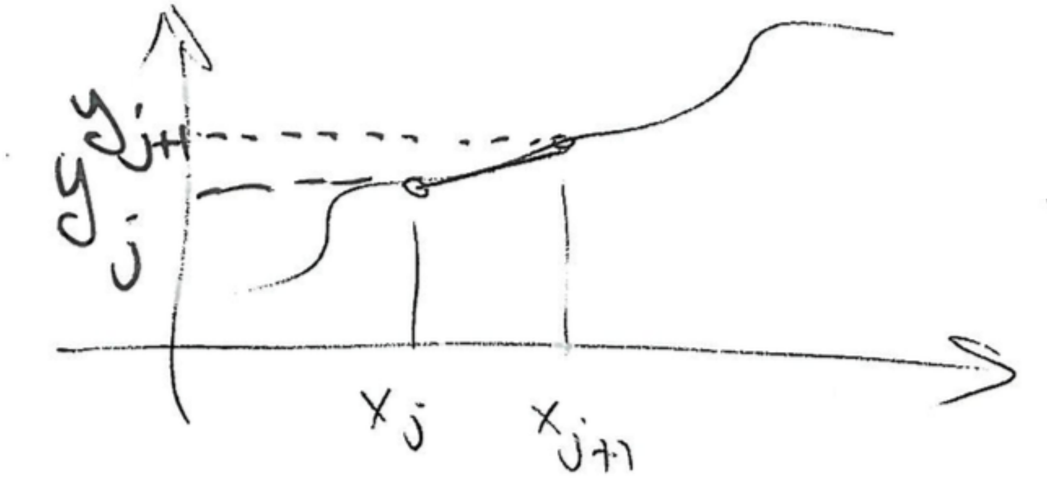

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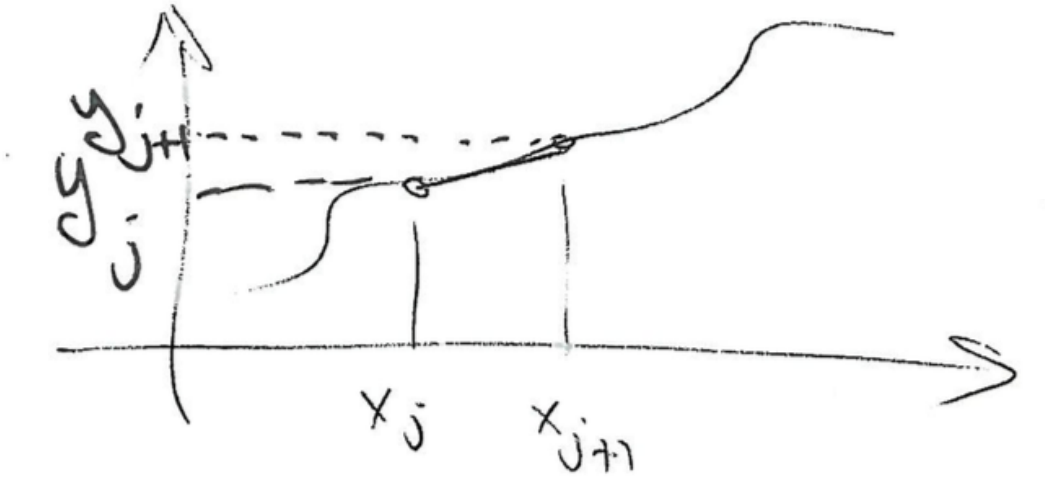

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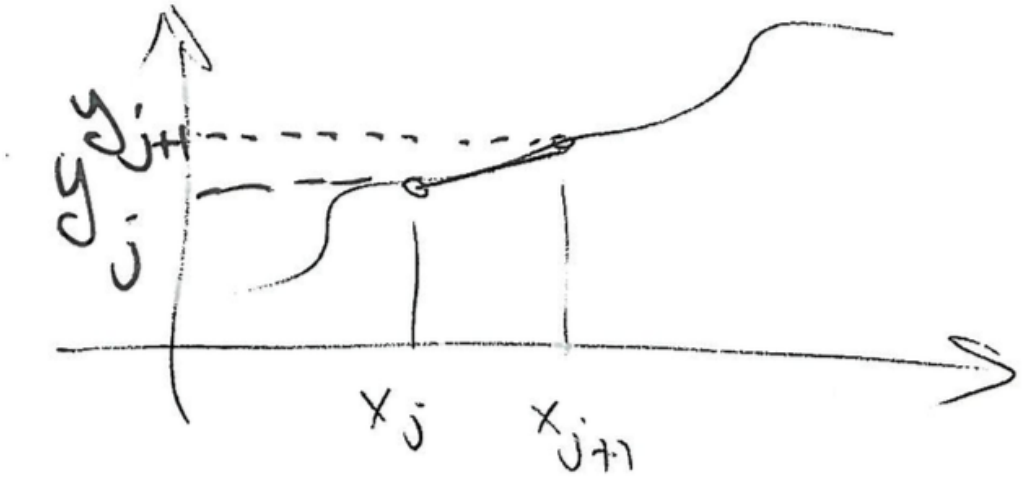

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where:
A=A(x), B=B(x),
C=C(x^3), D=D(x^3),
hence the name
cubic spline

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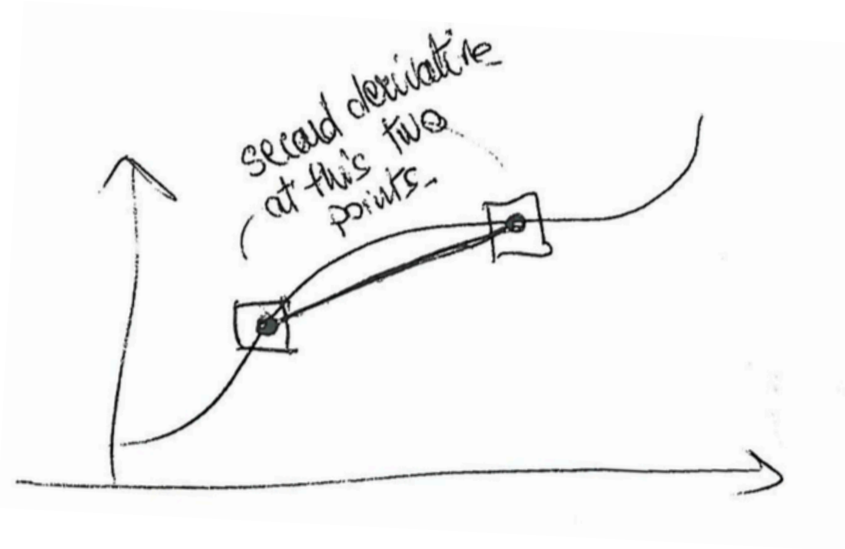

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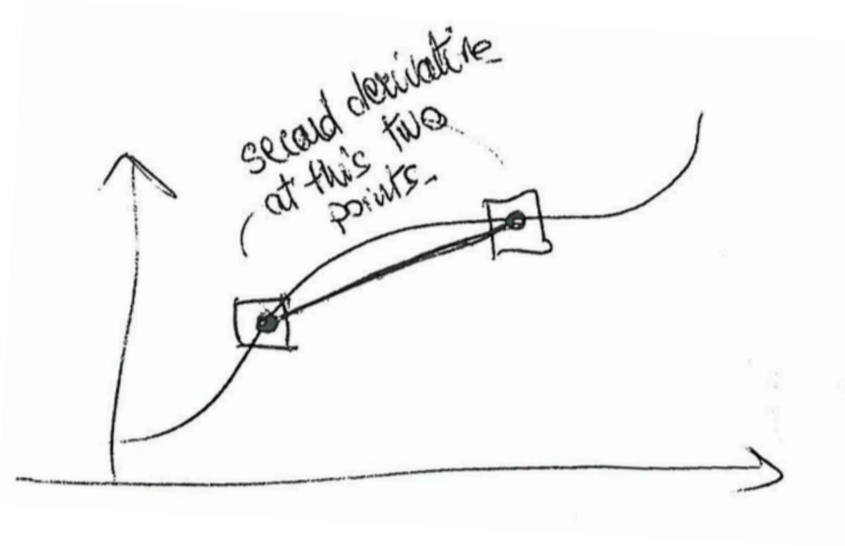

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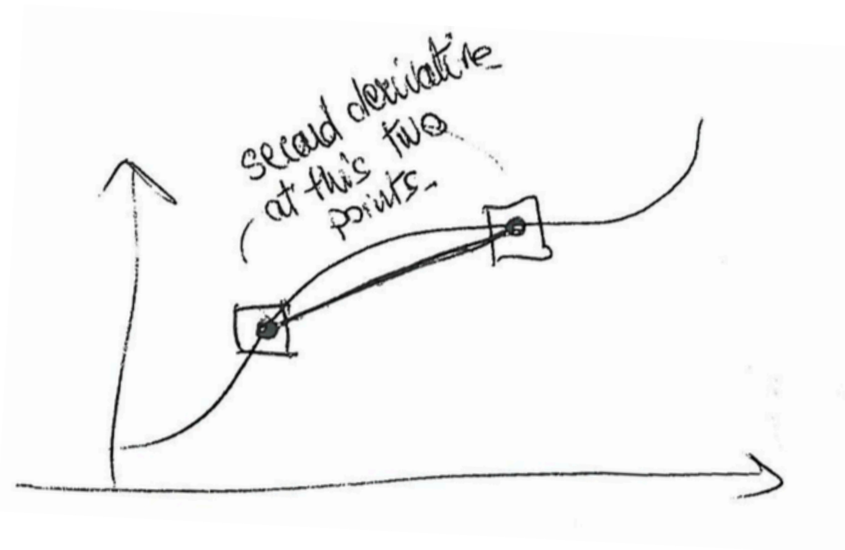

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🟦 $$\frac{dy}{dx} = y' = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{3A^2 - 1}{6}(x_{j+1} - x_j)y_j'' + \frac{3B^2 - 1}{6}(x_{j+1} - x_j)y_{j+1}''$$

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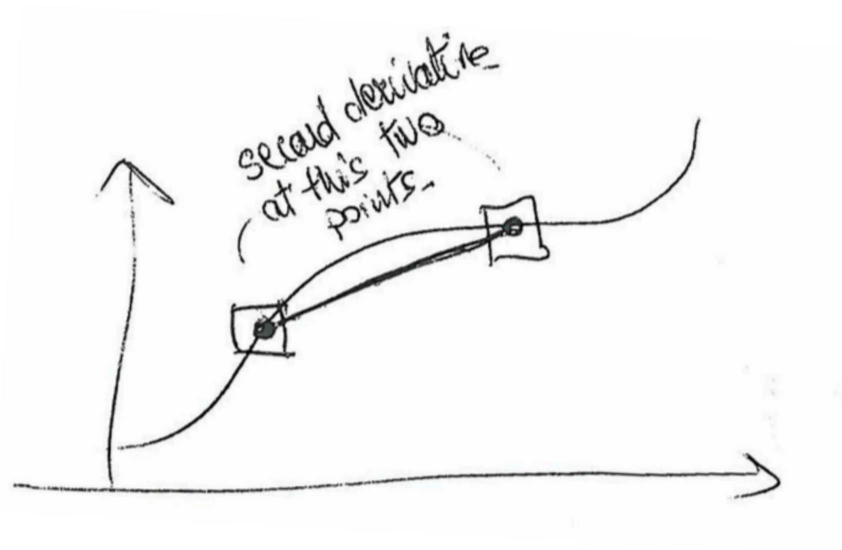

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$$\frac{dy}{dx} = y' = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{3A^2 - 1}{6}(x_{j+1} - x_j)y''_j + \frac{3B^2 - 1}{6}(x_{j+1} - x_j)y''_{j+1}$$

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$$\frac{d^2y}{dx^2} = y'' = -\frac{6AA'}{6}(x_{j+1} - x_j)y''_j + \frac{6BB'}{6}(x_{j+1} - x_j)y''_{j+1} = Ay''_j + By''_{j+1}$$

since $A(x_j) = 1 \quad A(x_{j+1}) = 0 \quad B(x_j) = 0 \quad B(x_{j+1}) = 1$

→ $C(x_j)=0$, $C(x_{j+1})=0$, $D(x_j)=0$, $D(x_{j+1})=0$, i.e., we can give any values we want to y_j'' and y_{j+1}''

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$$\frac{dy}{dx} = y'|_{x_j^-} = y'|_{x_j^+}$$

GLOBAL condition

Computed using x_{j-1} and x_j

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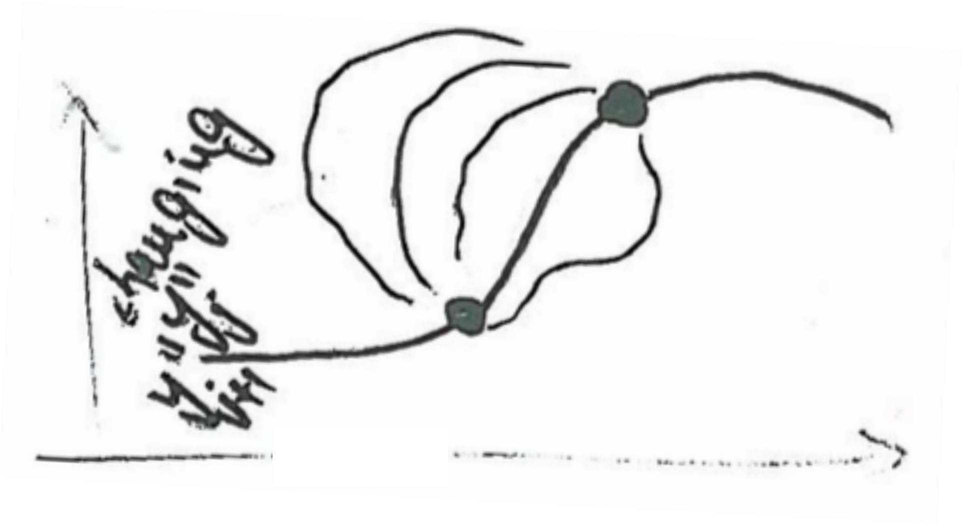
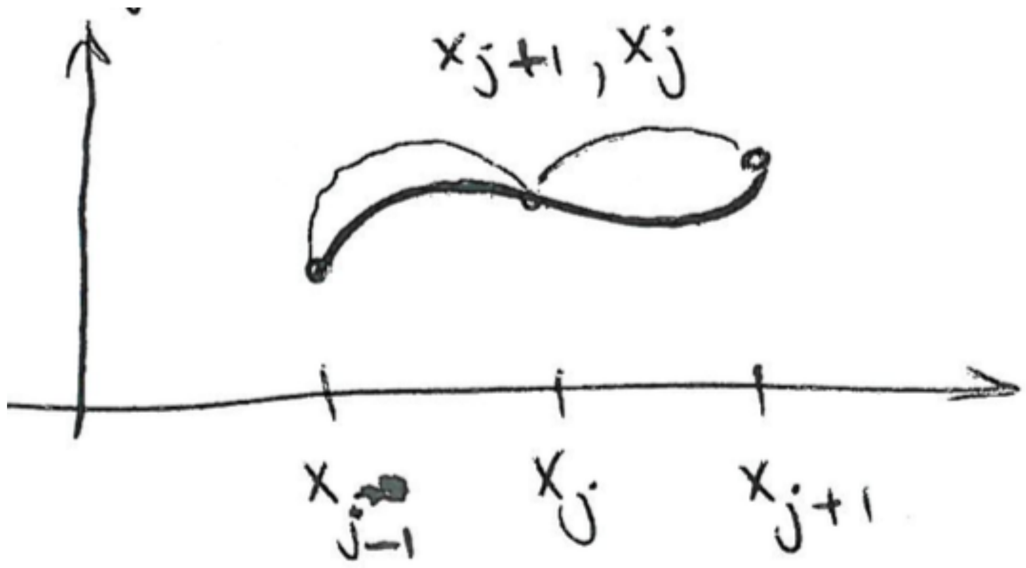


LOCALLY, for each $[x_j,x_{j+1}]$, I create a cubic, but this way I end up with different splines for each interval (local condition).



By setting the continuity of the first derivatives at x_j, I have a GLOBAL spline (cubic spline) and a smooth function (global).

$$y'|_{x_j^-} = \frac{y_j - y_{j-1}}{x_j - x_{j-1}} + \frac{1}{6}(x_j - x_{j-1})y''_{j-1} + \frac{1}{3}(x_j - x_{j-1})y''_j$$

$$y'|_{x_j^+} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{1}{3}(x_{j+1} - x_j)y''_j + \frac{1}{6}(x_{j+1} - x_j)y''_{j+1}$$

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$$\frac{x_j - x_{j-1}}{6}y''_{j-1} + \frac{x_{j+1} - x_{j-1}}{3}y''_j + \frac{x_{j-1} - x_j}{6}y''_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

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Repeating this also for j=2, …, N-1:

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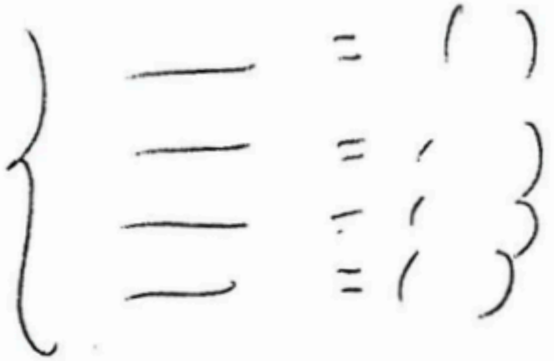
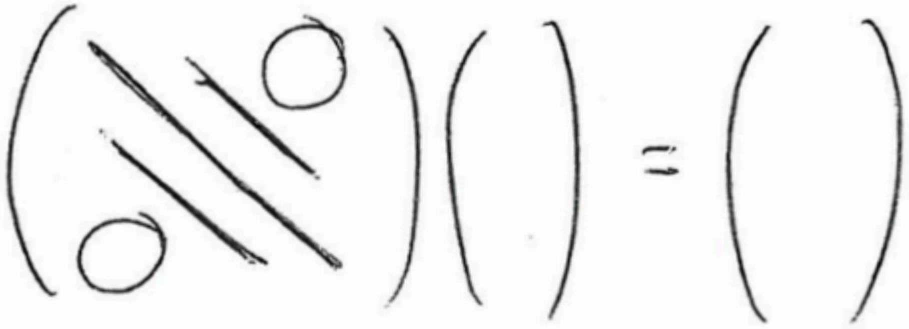


↔

tridiagonal system
(set of equations)



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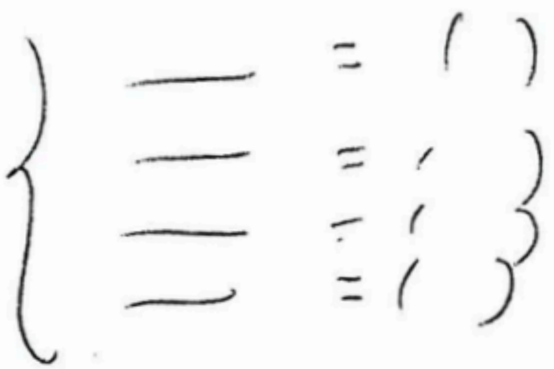
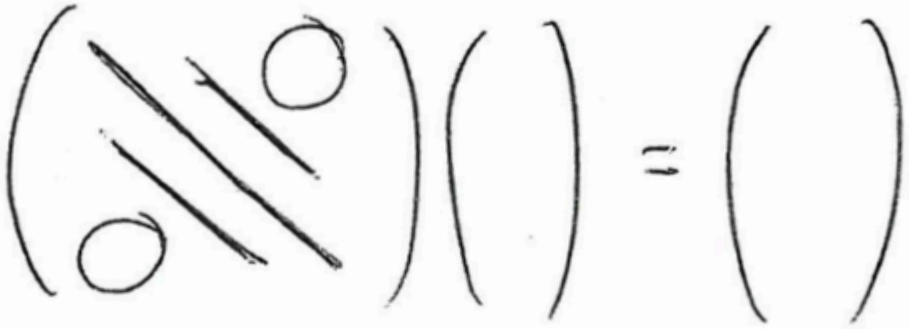
▲ 

↔ tridiagonal system (set of equations)

NxN tridiagonal matrix



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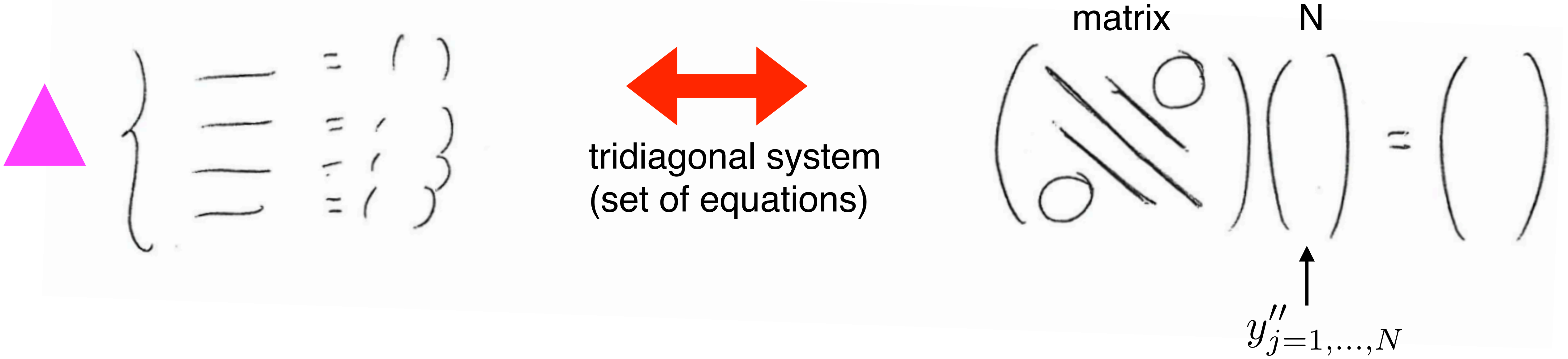

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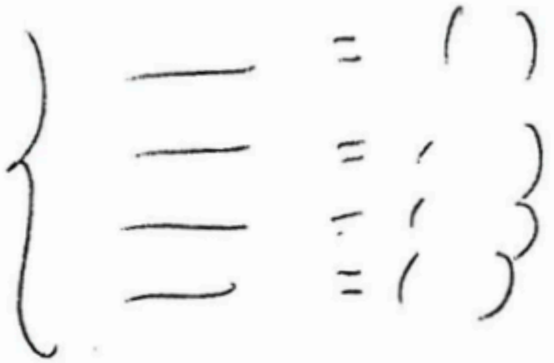
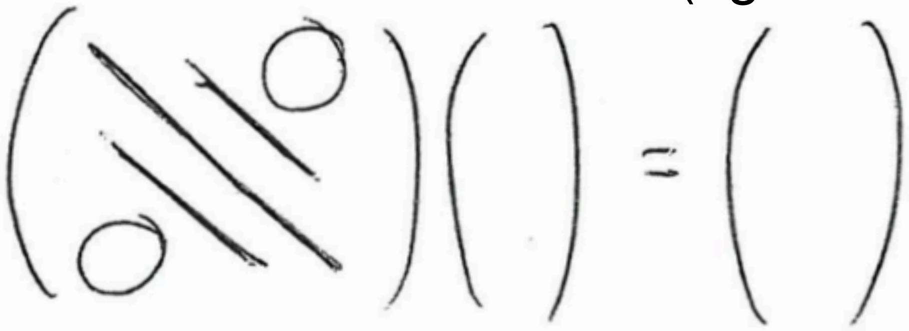


tridiagonal system
(set of equations)

NxN
tridiagonal
matrix

N

N
coefficients
(right hand)



$y''_{j=1,\ldots,N}$

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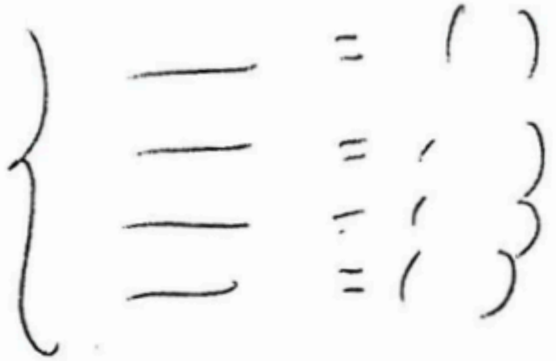
Repeating this also for j=2, …, N-1:

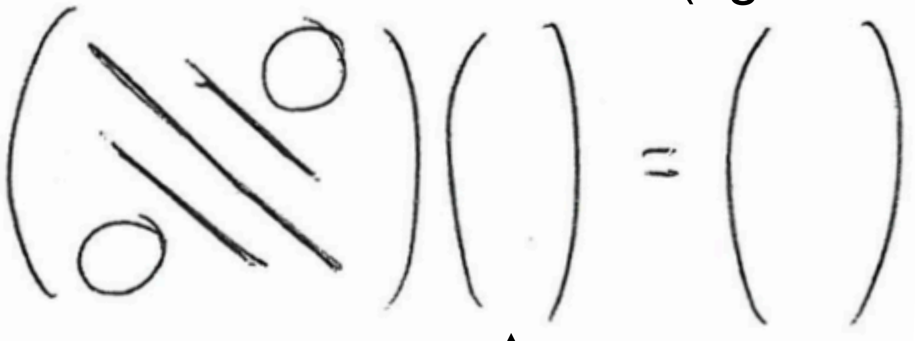


tridiagonal system
(set of equations)

NxN
tridiagonal
matrix

N

N
coefficients
(right hand)



$y''_{j=1,\dots,N}$

N-2 equations in N unknowns $y''_{j=1,\dots,N}$, i.e., each y”$_j$ is coupled only to its nearest neighbors at j+1 and j-1.

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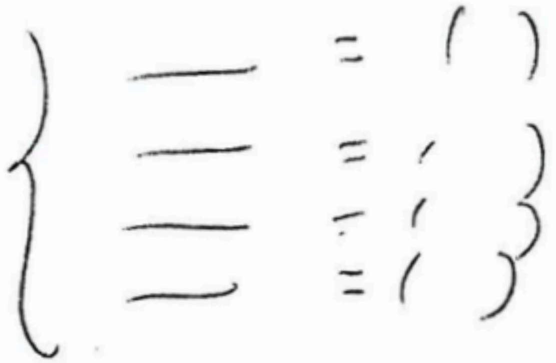

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NxN tridiagonal matrix

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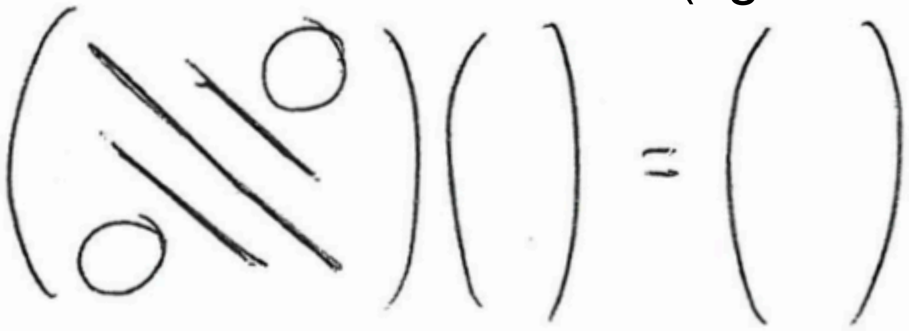

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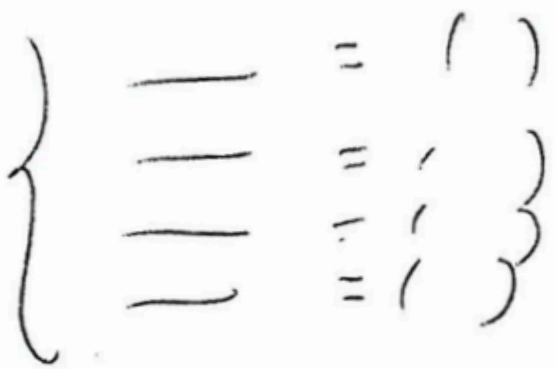

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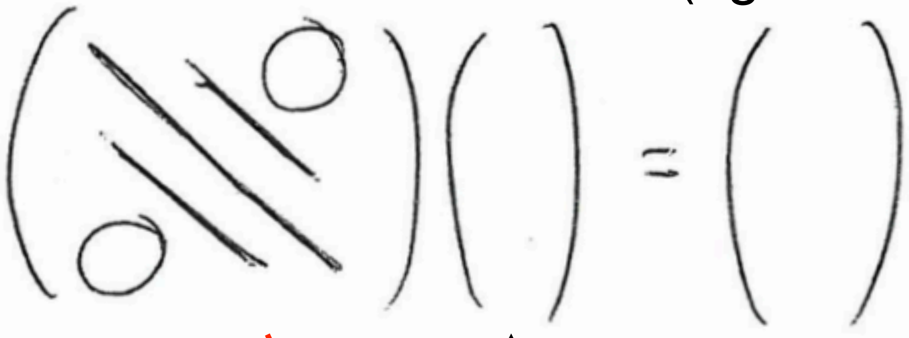

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Solve the NxN linear system of equations using Cramer’s rule

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IN GENERAL (except for the assignment), just use the function/routine in the library for linear, quadratic, cubic, or cubic spline interpolations.