ASTRONOMY 51 - SPRING 2022
Exercise Sheet 1
1A DUE by Friday, February 11, 2022 (300 points)
1B DUE by Friday, February 18, 2022 (300 points)

A. Root Finding

A completely ionized, homogeneous hydrogen plasma is irradiated by X-ray and the electron scattering optical depth is very large. The electrons at temperature T_e are exchanging energy with protons at temperature T_p via Coulomb collisions and with photons at temperature T_γ via Compton scattering. The magnetic field and all other radiative processes are negligible. Under these conditions the energy balance equation for the electron plasma can be written as:

$$f(T_\mathrm{e}) = Q\frac{U}{\rho c^2}(T_\mathrm{e} - T_\gamma) - \frac{\Lambda}{c^3}\left(\frac{T_\mathrm{p}}{T_\mathrm{e}} - 1\right)\frac{1}{\sqrt{T_\mathrm{e}}} = 0, \qquad (1)$$

where $Q \equiv 4k_\mathrm{es}K_\mathrm{B}/m_\mathrm{e}c^2 = 2.7\times10^{-10}$ cm^{2}g^{-1}K^{-1} (with k_es=0.4 cm^{2}g^{-1} being the electron scattering opacity), $U/(\rho c^2)$ is the ratio between the radiation energy density and the rest-mass energy density of the electron-proton plasma, and Λ=4.4$\times10^{30}$ is a constant.

(i) Consider the case in which the electrons are heated by Coulomb collisions with the protons at temperature T_p=10^9K and cooled via inverse Compton scattering with photons at temperature T_γ=10^7K. Solve Eq. 1 with $U/(\rho c^2)$=1 and find the equilibrium temperature for the electrons T_e. The bracketing interval can be found after plotting Eq. 1 as a function of T_e.

(ii) Consider now the case in which the electrons are cooled by Coulomb collisions with the protons at temperature T_p=10^7K and heated via Compton scattering with photons at temperature T_γ=10^9K. Solve Eq. 1 with $U/(\rho c^2)$=8$\times10^{-5}$ and find the equilibrium temperature for the electrons using as bracketing interval T_1=10^7K and T_2=10^9K

NOTE:

(i) Use both the bisection and the Newton-Raphson methods concentrating on the number of iterations necessary to reach the desired accuracy of ϵ=10^{-7}, i.e., quantify the number of iterations in both methods and compare them.

(ii) To make results comparable, use as speed of light c =2.99$\times10^{10}$ cm s^{-1}

B. Polynomial Interpolation

Consider the function

$$y(x) = 3 + 200x - 30x^2 + 4x^3 - x^4, \quad (2)$$

in the interval I: $x \in [-10, 10]$.

(i) Find its roots in I if they exists (check whether roots exist by plotting Eq. 2). Use, as bracketing intervals, e.g., $y_1 = -1$ and $y_2 = 1$ for one root, and $y_3 = 4$ and $y_4 = 6$ for the other root.

(ii) Interpolate the value of the function at $x = -5$ and $x = 5$ using a linear, a quadratic, and a cubic interpolating polynomial. Calculate the error made in each case. Write the adopted input values, the results, and the errors in tables for each interpolating methods. For the linear polynomial, use $x_1 = -6$ and $x_2 = -4$ for $x = -5$, and $x_1 = 4$ and $x_2 = 6$ for $x = 5$. For the quadratic polynomial, use $x_1 = -6$, $x_2 = -4$, and $x_3 = -3$ for $x = -5$, and $x_1 = 4$, $x_2 = 6$, and $x_3 = 7$ for $x = 5$. For the cubic polynomial, use $x_1 = -7$, $x_2 = -6$, $x_3 = -4$, and $x_4 = -3$ for $x = -5$, and $x_1 = 3$, $x_2 = 4$, $x_3 = 6$, and $x_4 = 7$ for $x = 5$.

(iii-BONUS) Interpolate the value of the function at $x = 12$ using now a cubic spline. Compare with the results obtained before. Write the adopted input values, the result, and the error in a table. For the cubic spline, use $x_1 = 11$, $x_2 = 11.5$, $x_3 = 12.5$, and $x_4 = 13$ for $x = 12$, and use Cramer's rule to solve the tridiagonal set of equations.