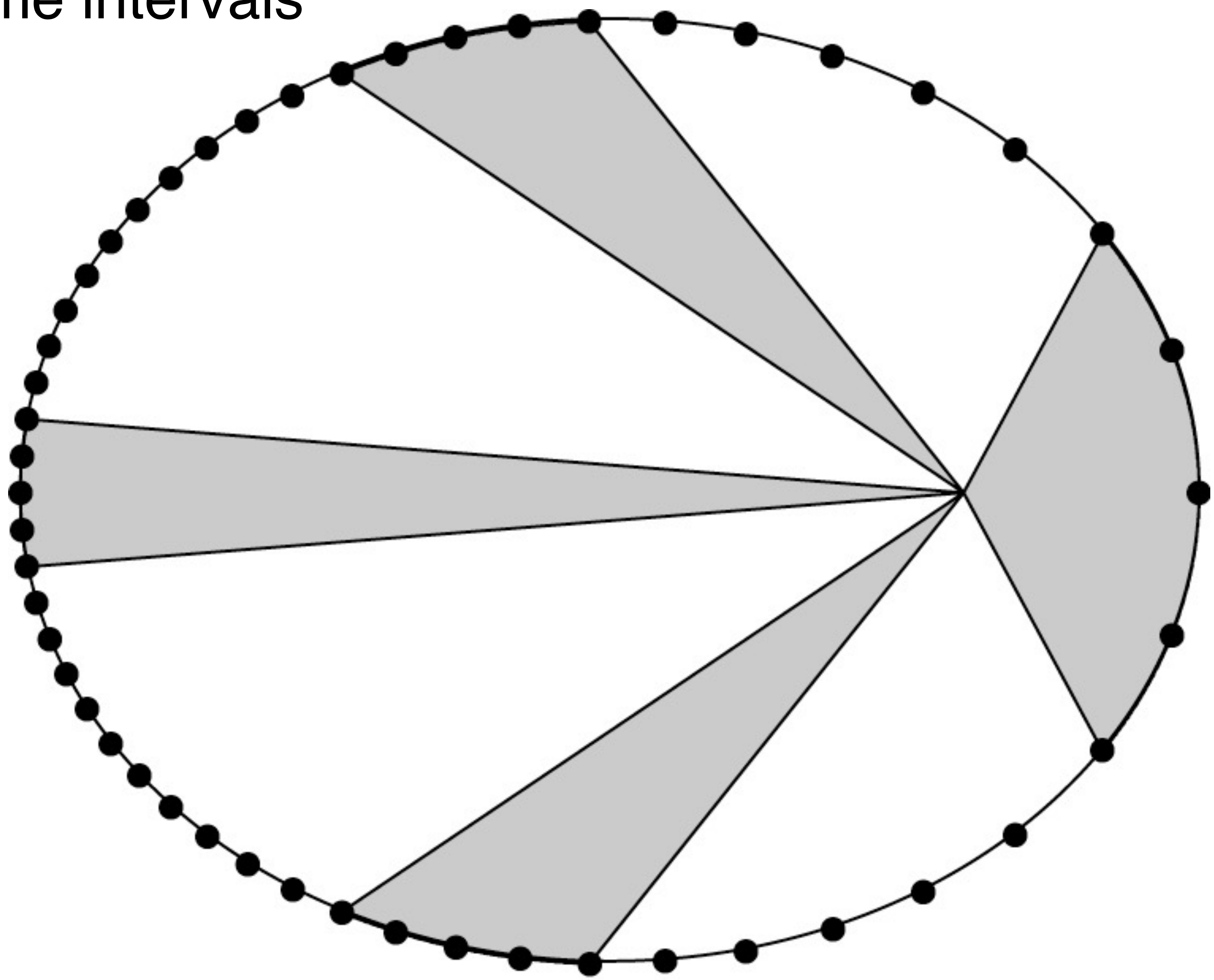
Celestial Mechanics

Kepler's Laws:

I. A planet orbits the Sun in an ellipse, with the Sun at one focus of the ellipse

II. A line connecting a planet to the Sun sweeps out equal areas in equal time intervals

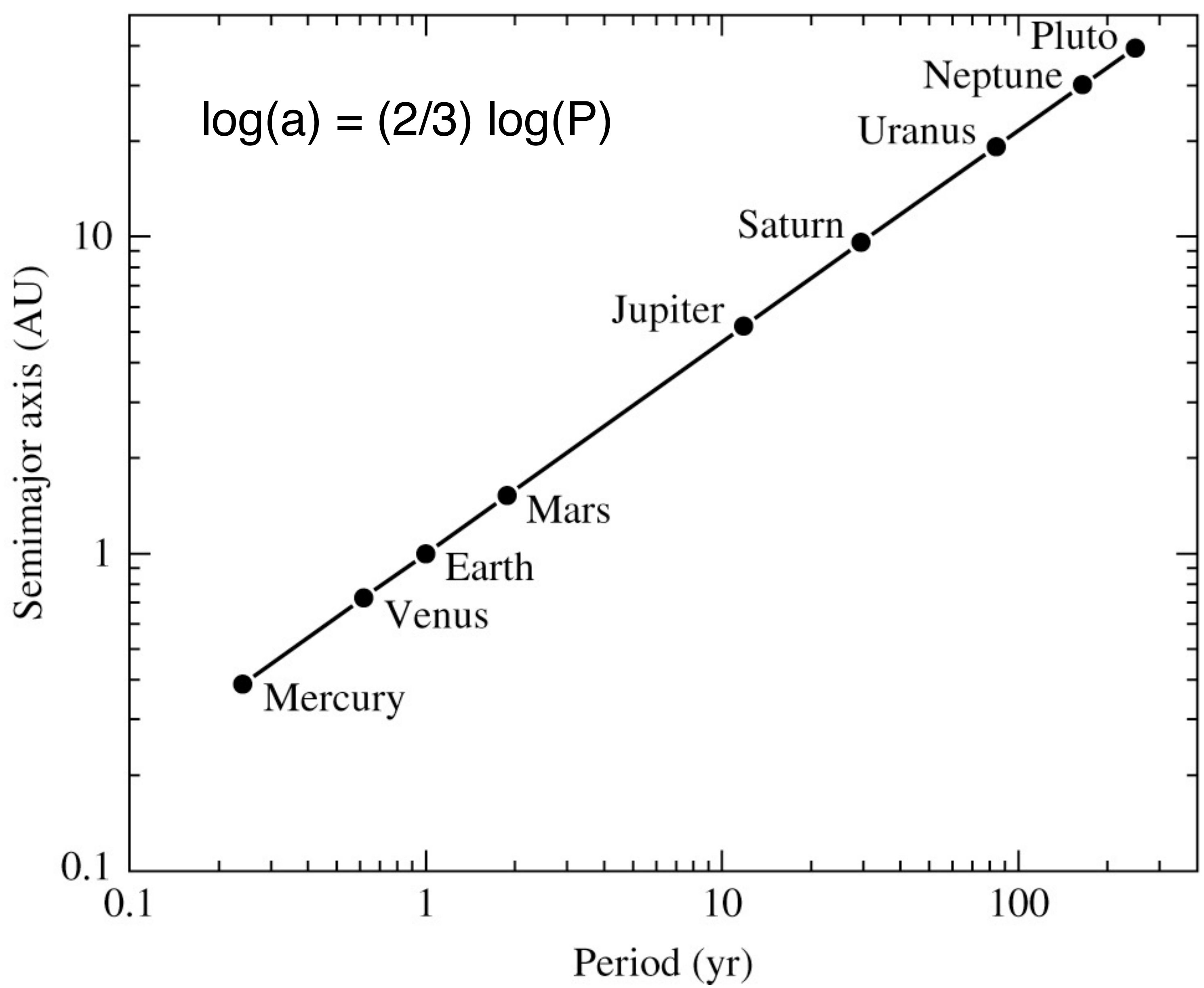


Kepler's Laws:

I. A planet orbits the Sun in an ellipse, with the Sun at one focus of the ellipse

II. A line connecting a planet to the Sun sweeps out equal areas in equal time intervals

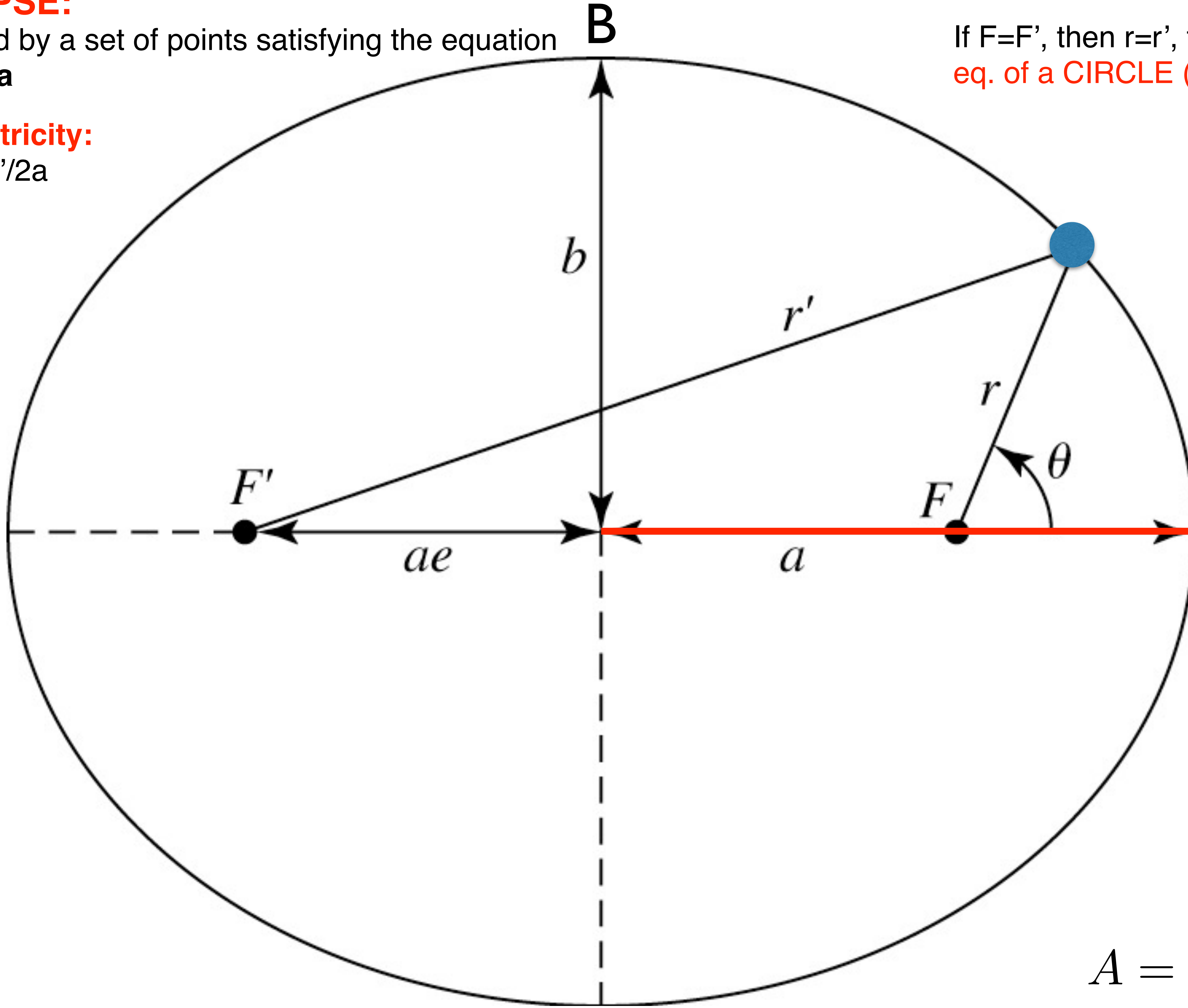
III. $P[yr]^2 = a[AU]^3$, with P orbital period of the planet, a average distance of the planet from the Sun



ELLIPSE:
defined by a set of points satisfying the equation
r+r'=2a

Eccentricity:
e = FF'/2a
0<e<1

If F=F', then r=r', then r=a
eq. of a CIRCLE (e=0)



$$A = \pi a b$$

ELLIPSE:

defined by a set of points satisfying the equation

r+r'=2a

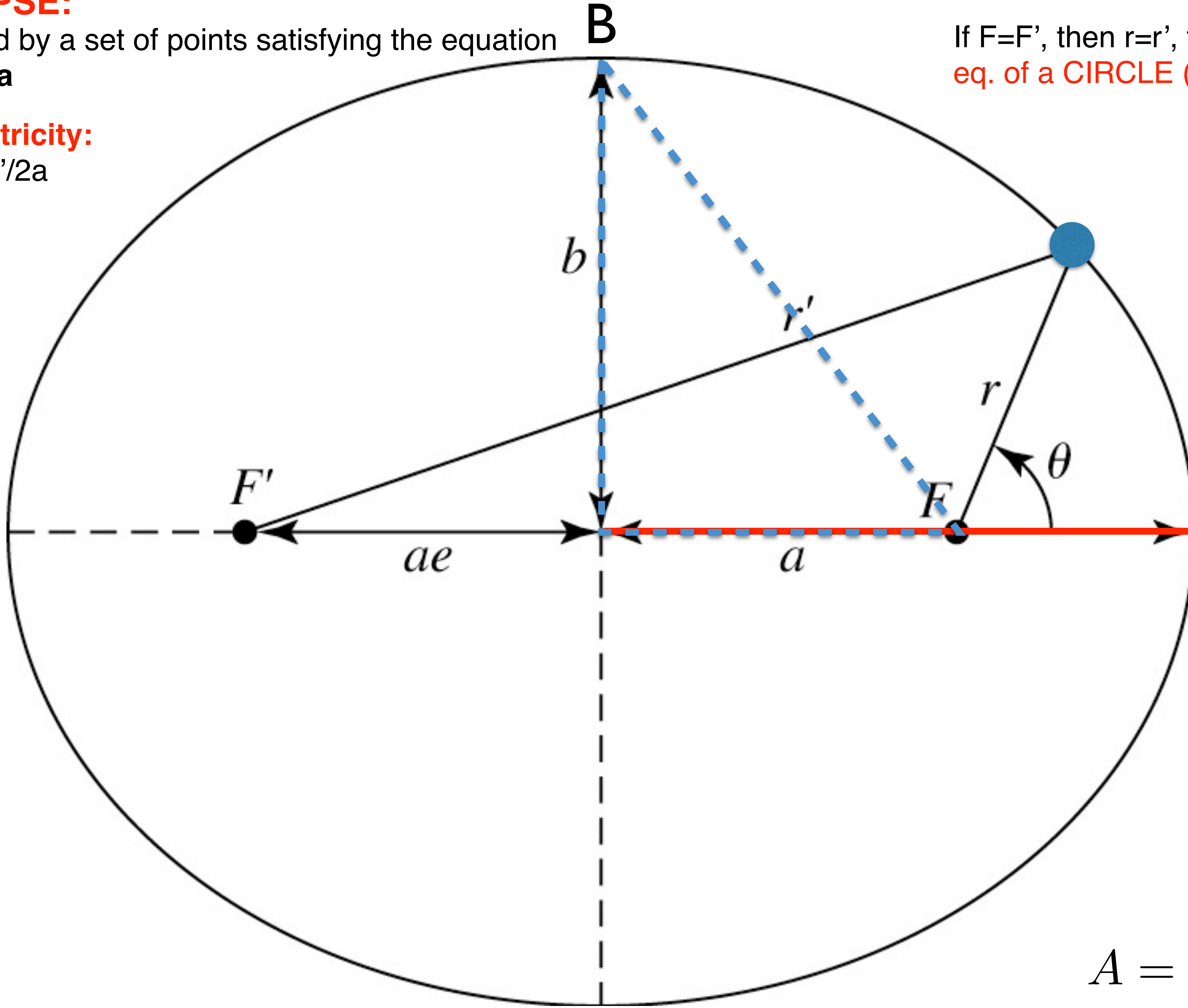
Eccentricity:

e = FF'/2a

$0<e<1$

If F=F', then r=r', then r=a

eq. of a CIRCLE (e=0)



$$A = \pi ab$$

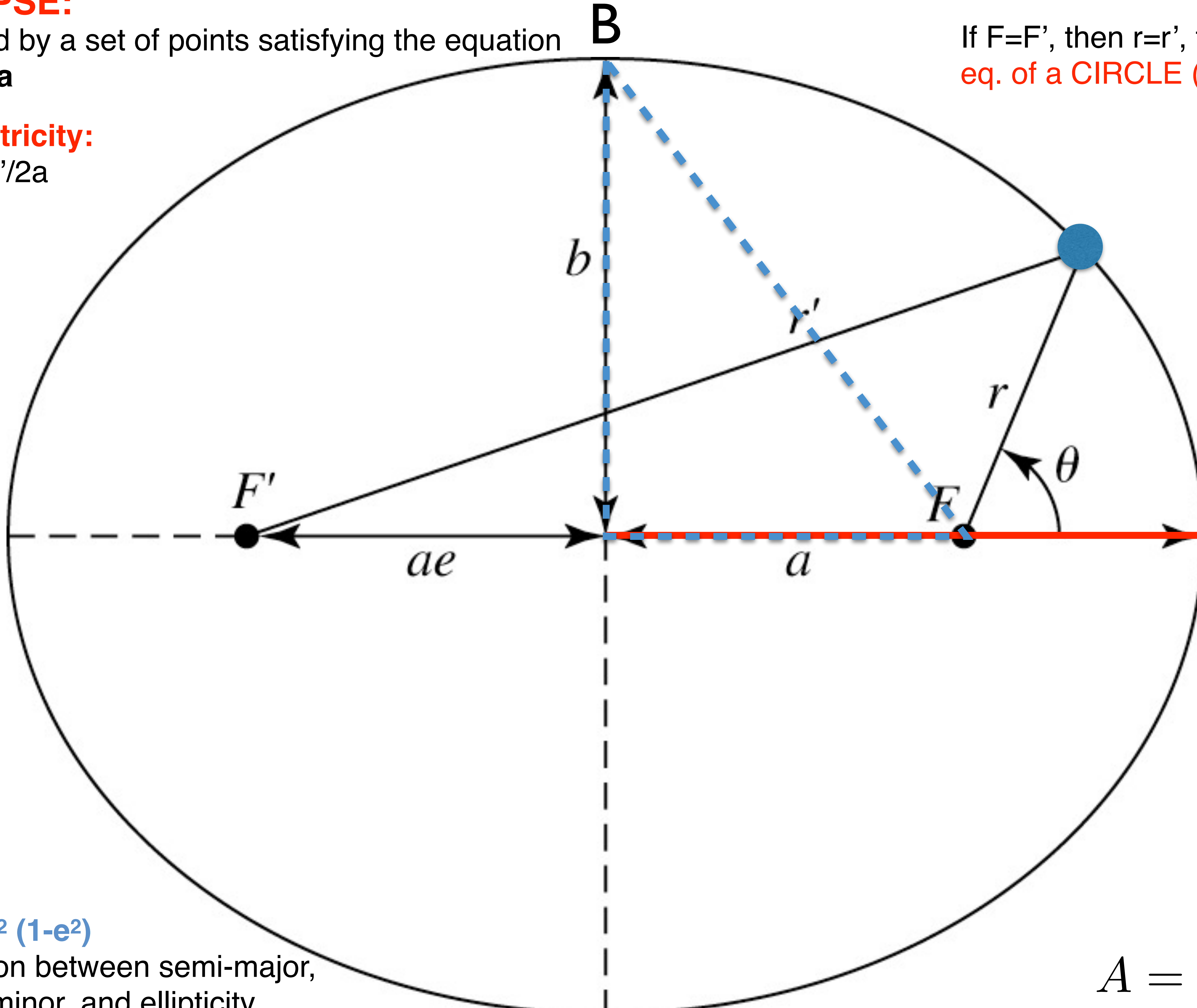
ELLIPSE:

defined by a set of points satisfying the equation
r+r'=2a

Eccentricity:

e = FF'/2a
0<e<1

If F=F', then r=r', then r=a
eq. of a CIRCLE (e=0)



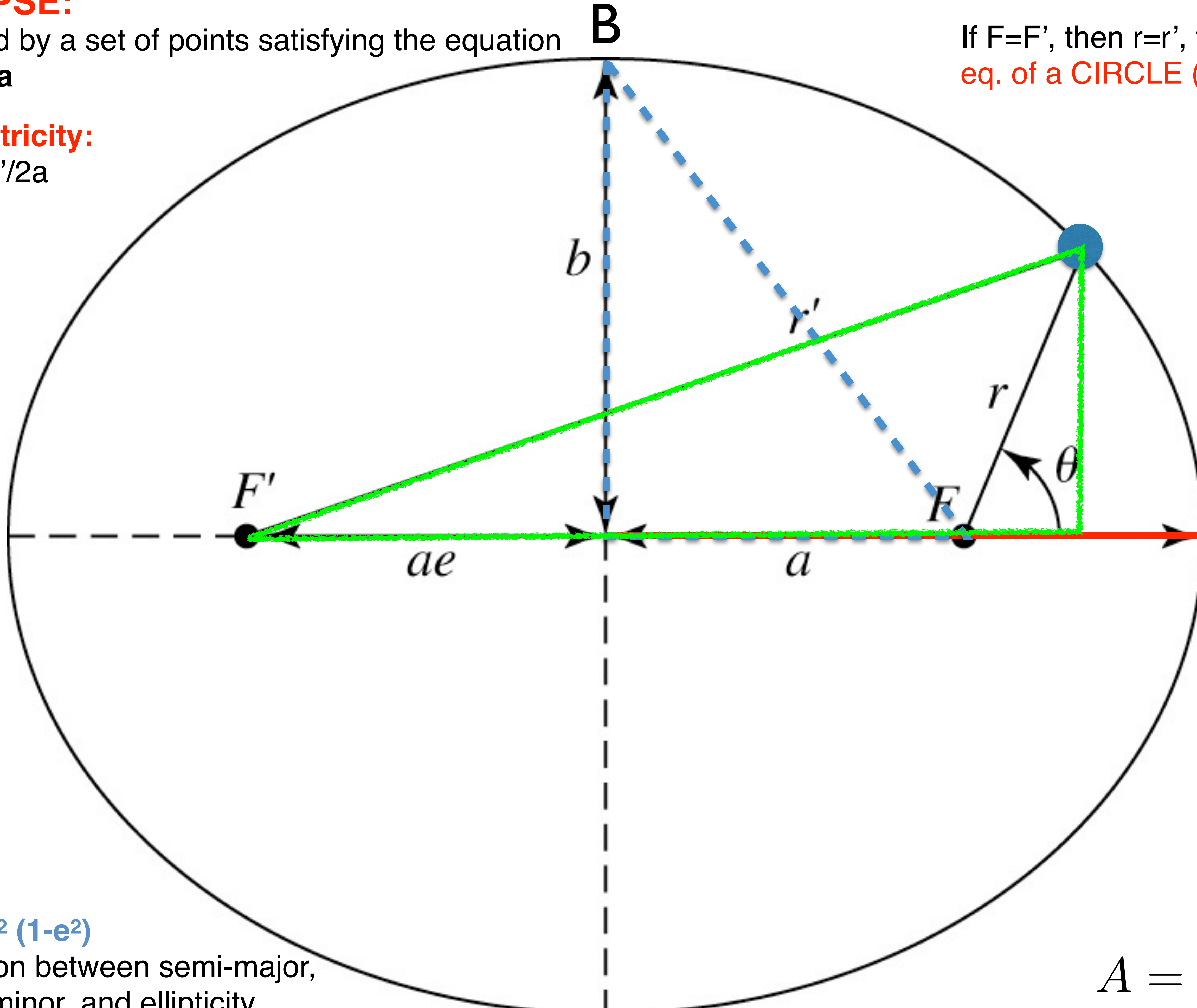
$b^2 = a^2 (1-e^2)$
Relation between semi-major, semi-minor, and ellipticity

$$A = \pi a b$$

ELLIPSE:
defined by a set of points satisfying the equation
r+r'=2a

Eccentricity:
e = FF'/2a
0<e<1

If F=F', then r=r', then r=a
eq. of a CIRCLE (e=0)



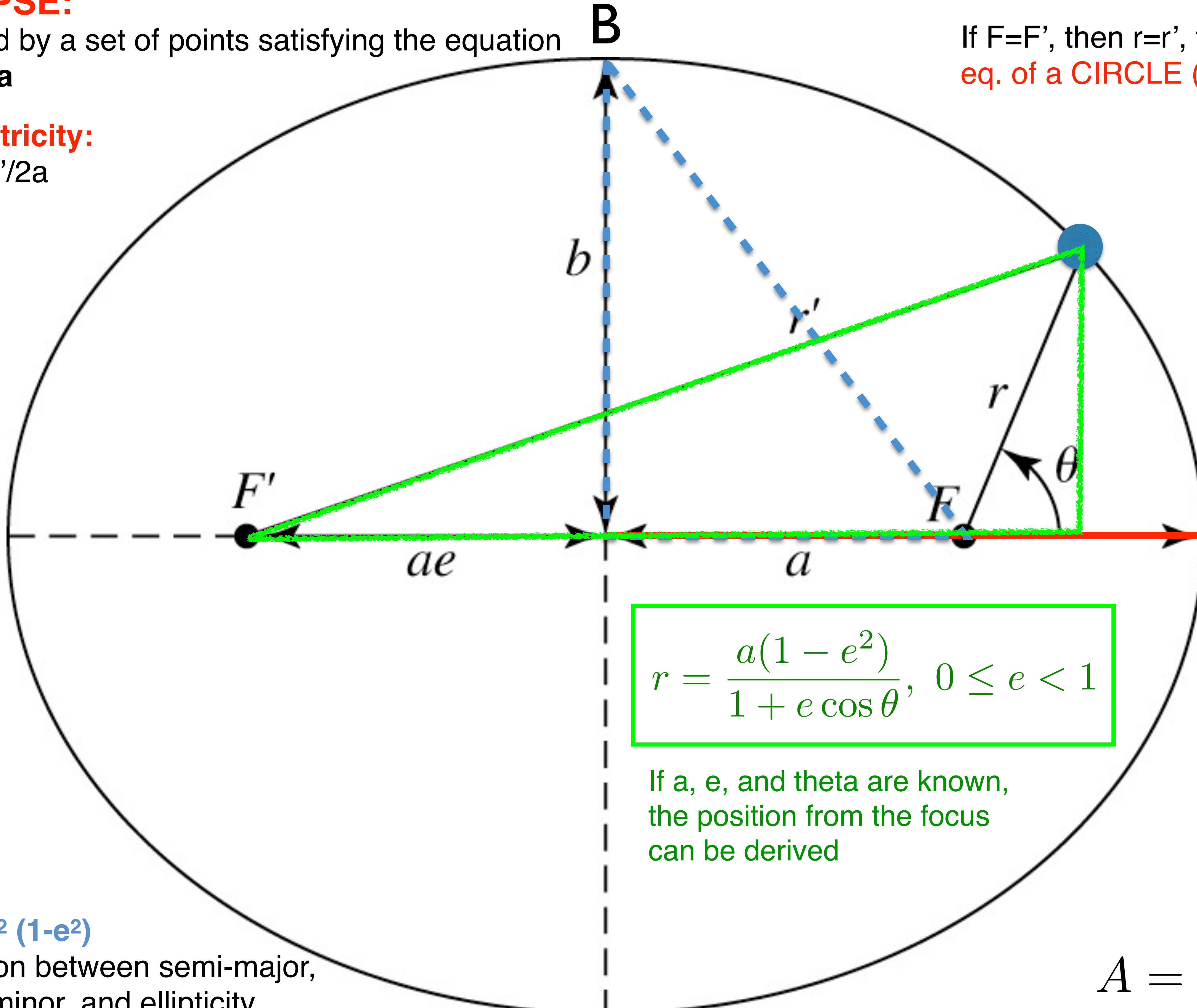
$b^2 = a^2 (1-e^2)$
Relation between semi-major, semi-minor, and ellipticity

$$A = \pi ab$$

ELLIPSE:
defined by a set of points satisfying the equation
r+r'=2a

Eccentricity:
e = FF'/2a
0<e<1

If F=F', then r=r', then r=a
eq. of a CIRCLE (e=0)



$$r = \frac{a(1-e^2)}{1+e\cos\theta}, \quad 0 \le e < 1$$

If a, e, and theta are known, the position from the focus can be derived

$b^2 = a^2 (1-e^2)$
Relation between semi-major, semi-minor, and ellipticity

$$A = \pi ab$$

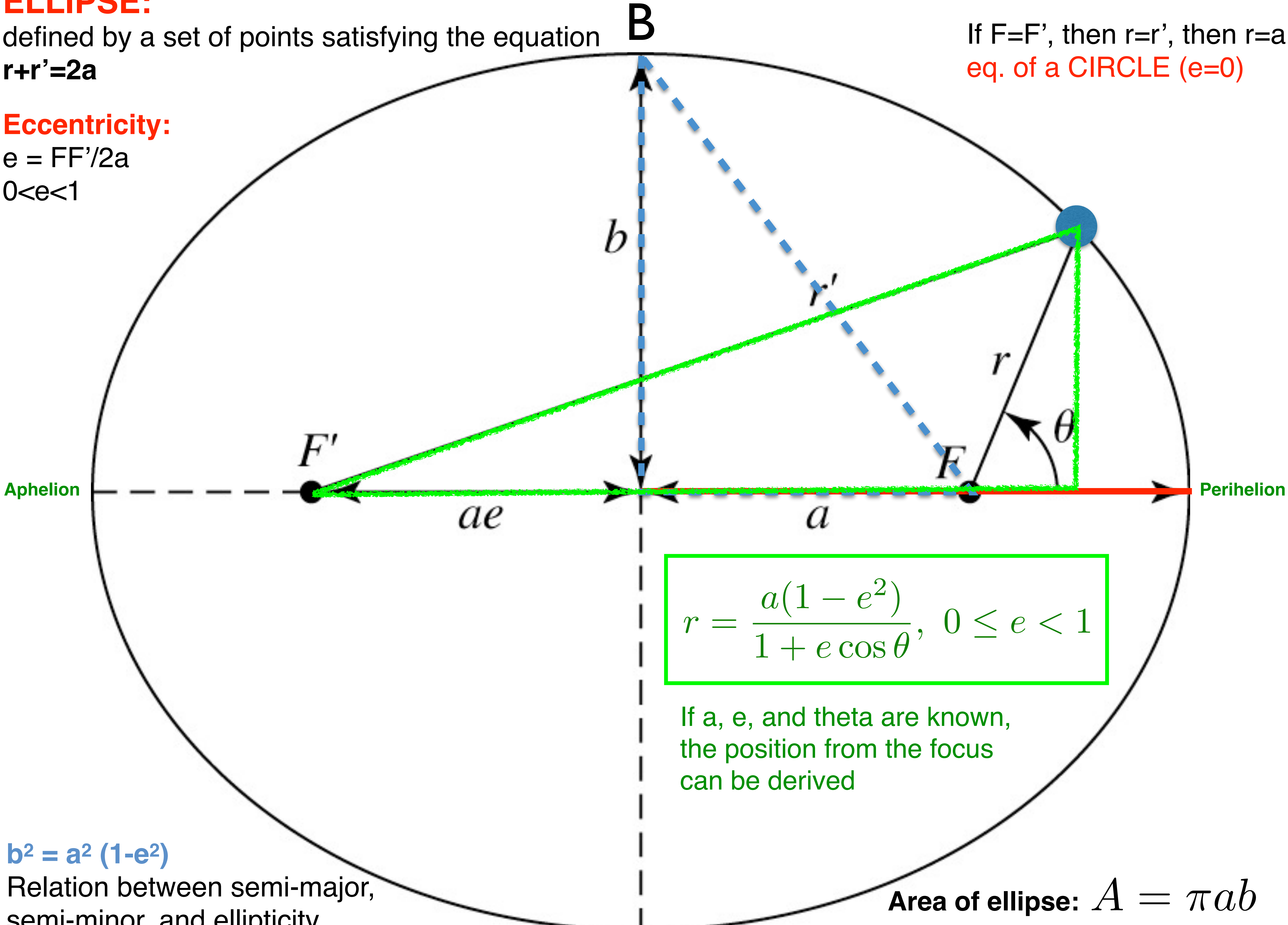
ELLIPSE:

defined by a set of points satisfying the equation
r+r'=2a

Eccentricity:

e = FF'/2a
0<e<1

If F=F', then r=r', then r=a
eq. of a CIRCLE (e=0)



$$r = \frac{a(1-e^2)}{1+e\cos\theta}, \quad 0 \leq e < 1$$

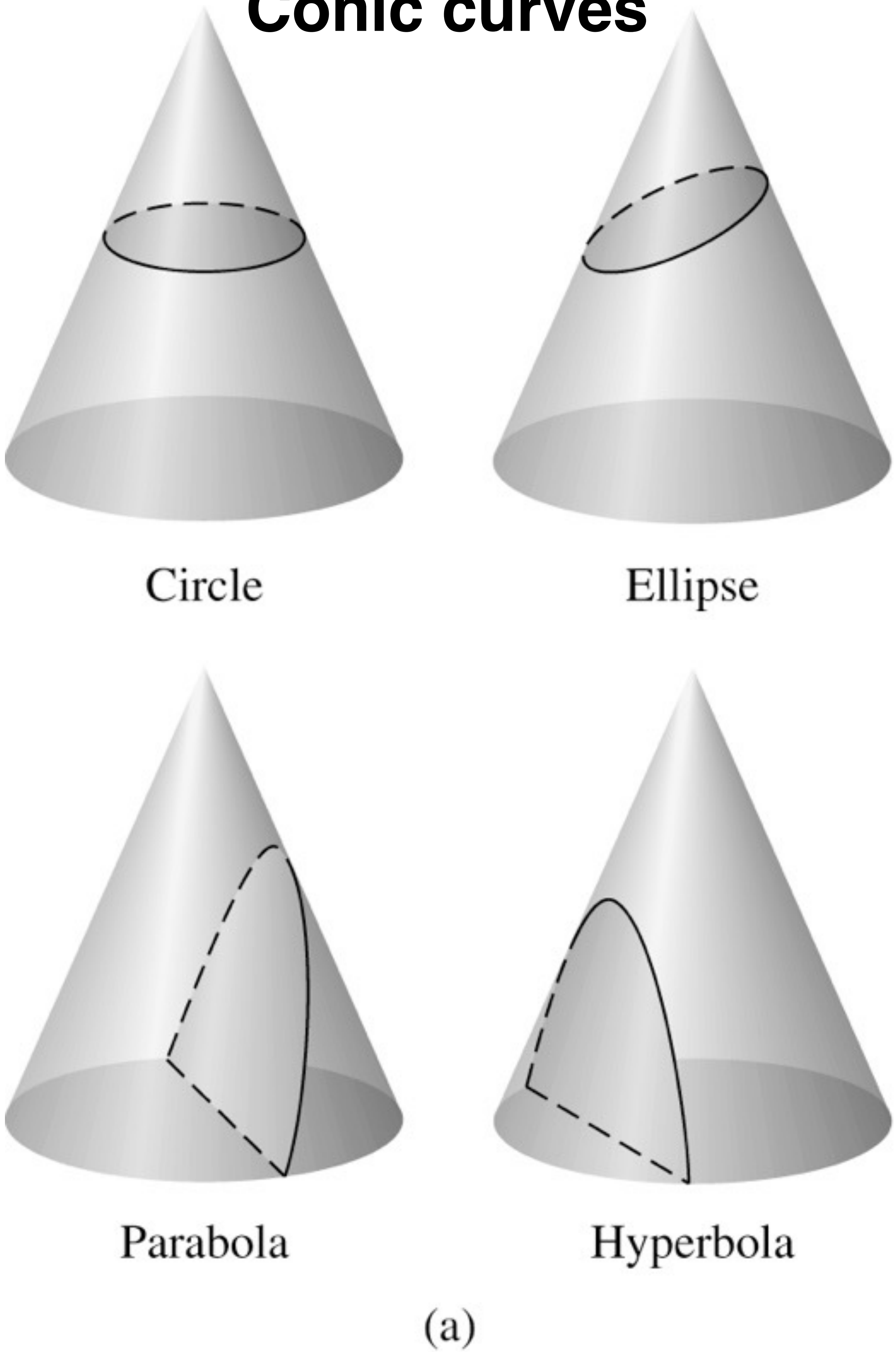
If a, e, and theta are known, the position from the focus can be derived

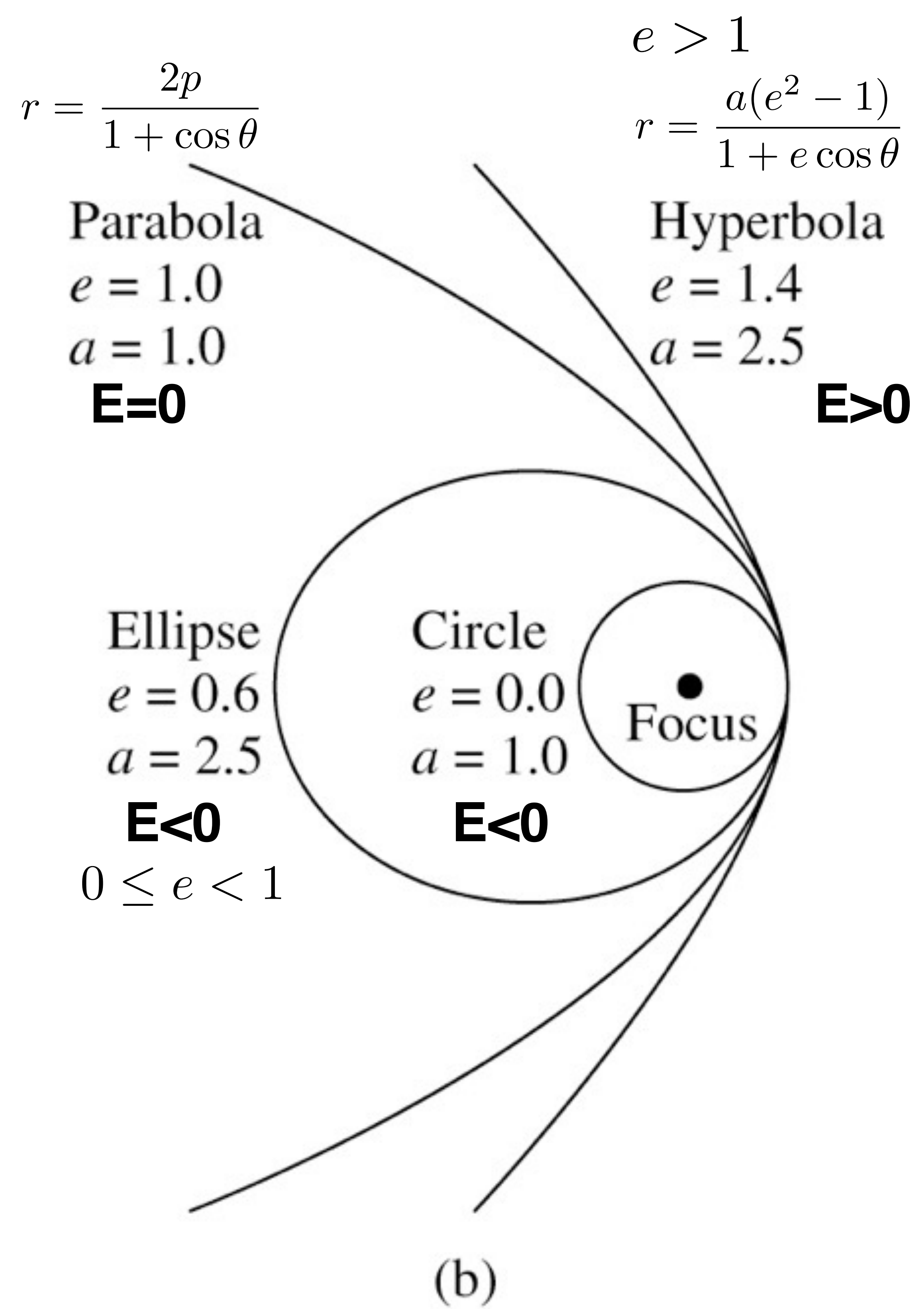
$b^2 = a^2 (1-e^2)$
Relation between semi-major, semi-minor, and ellipticity

Area of ellipse: $A = \pi ab$

Conic curves

E total energy
E<0 —> bound system
E>0 —> unbound system





Each type of conic section is related to a specific form of celestial motion

Newton's Laws:

I. An object at rest will remain at rest and an object in motion will remain in motion in a straight line at a constant speed unless acted upon by an external force (law of inertia)

= the momentum of an object remains constant unless it experiences an external force

$$\mathbf{p} = m\mathbf{v}$$

Newton's Laws:

I. An object at rest will remain at rest and an object in motion will remain in motion in a straight line at a constant speed unless acted upon by an external force (law of inertia)

= the momentum of an object remains constant unless it experiences an external force

II. The net force (the sum of all forces) acting on an object is proportional to the object's mass and its resultant acceleration

$$\mathbf{F}_{\mathrm{net}} = \sum_{i=1}^{n} \mathbf{F}_{\mathrm{i}} = m\mathbf{a}$$

$$m = \text{constant}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

→

$$\mathbf{F}_{\mathrm{net}} = \frac{d\mathbf{p}}{dt}$$

The net force on an object is equal to the time rate of change of its momentum

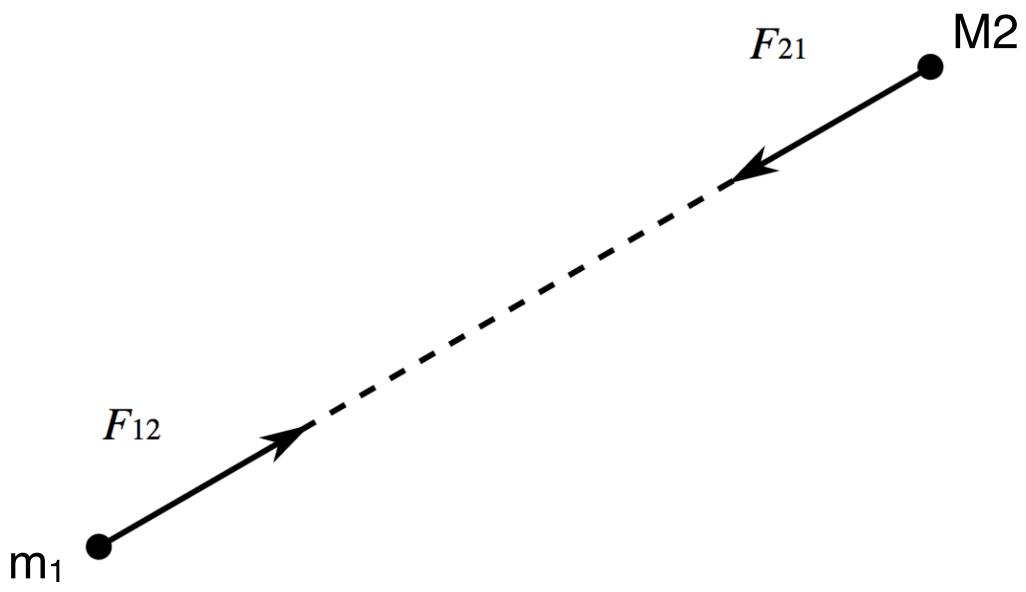
Newton's Laws:

I. An object at rest will remain at rest and an object in motion will remain in motion in a straight line at a constant speed unless acted upon by an external force (law of inertia)

= the momentum of an object remains constant unless it experiences an external force

II. The net force (the sum of all forces) acting on an object is proportional to the object's mass and its resultant acceleration

III. For every action there is an equal and opposite reaction



$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

Action and reaction are forces acting on DIFFERENT objects

Newton's Laws:

I. An object at rest will remain at rest and an object in motion will remain in motion in a straight line at a constant speed unless acted upon by an external force (law of inertia)

= the momentum of an object remains constant unless it experiences an external force

II. The net force (the sum of all forces) acting on an object is proportional to the object's mass and its resultant acceleration

III. For every action there is an equal and opposite reaction

Newton's Law of Universal Gravitation:

$$F = G\frac{Mm}{r^2}, \quad G = 6.673 \times 10^{-11} \ \frac{\mathrm{N\ m^2}}{\mathrm{kg^2}}$$

Work and Energy

The amount of energy (the work) needed to raise an object of mass m to a height h against a gravitational force is equal to the change in the potential energy of the system:

Work Integral: $$U_{\mathrm{f}} - U_{\mathrm{i}} = \Delta U \equiv -\int_{\mathbf{r}_{\mathrm{i}}}^{\mathbf{r}_{\mathrm{f}}} \mathbf{F} \cdot d\mathbf{r}$$

For the gravitational force on m being due to a mass M located at the orgin: $\mathbf{F} \cdot d\mathbf{r} = -F dr$

$$\Delta U = \int_{r_{\mathrm{i}}}^{r_{\mathrm{f}}} G \frac{mM}{r^2} dr$$

$$\Rightarrow \quad U = -G\frac{Mm}{r} \quad (\text{where } \mathrm{U_f} = 0 \text{ at } \mathrm{r_f} = \infty)$$

Work and Energy

Work must be performed on a massive object if its speed is to be changed:

$$W \equiv -\Delta U = \ldots = \frac{1}{2}mv_{\mathrm{f}}^2 - \frac{1}{2}mv_{\mathrm{i}}^2$$

$$K \equiv \frac{1}{2}mv^2$$ Kinetic energy of an object

➡ The work done on the particle results in an equivalent change in the particle's kinetic energy (conservation of energy)

$$W = \Delta K = \frac{1}{2}m(v_{\mathrm{f}}^2 - v_{\mathrm{i}}^2)$$

$$E = K + U = \frac{1}{2}mv^2 - G\frac{Mm}{r}$$

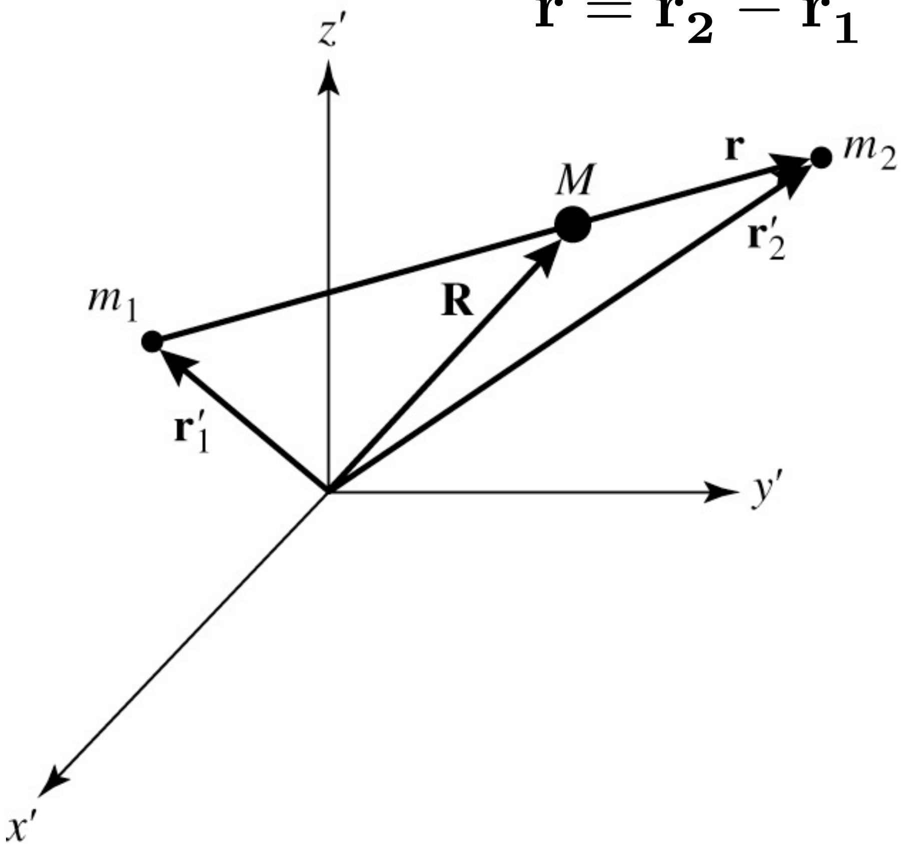
Energy of a particle of mass m with velocity **v** at a distance r from the center of a larger mass M

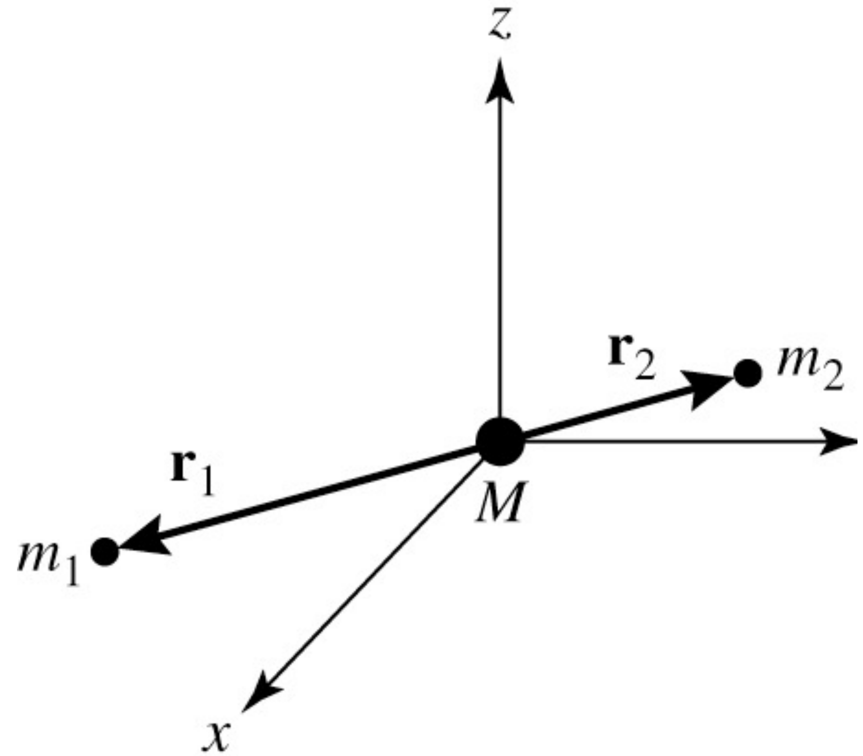
$\mathrm{At\ r} = \infty,\ \mathrm{v} = 0$ ➡ $E = 0$ ➡ $\frac{1}{2}mv^2 = G\frac{Mm}{r}$

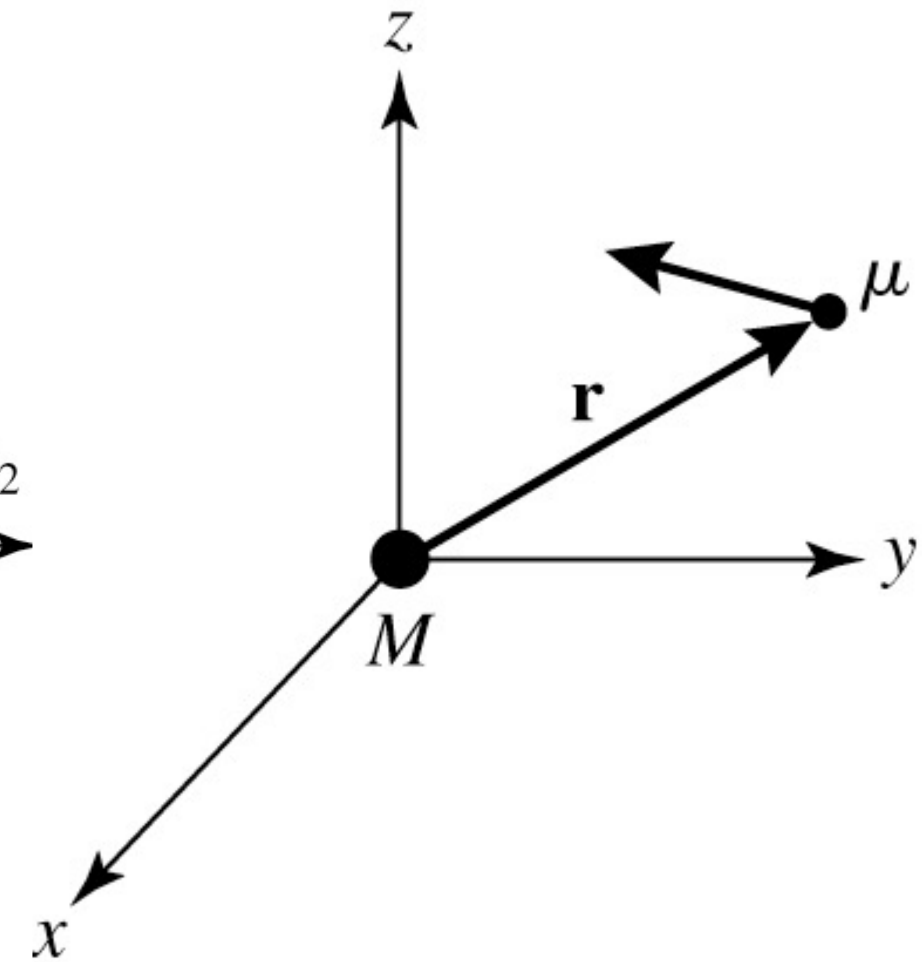
➡ $v_{\mathrm{esc}} = \sqrt{2GM/r}$ The escape velocity is independent on m

Center of mass

$\mathbf{r} = \mathbf{r_2'} - \mathbf{r_1'}$ displacement vector







$$\mathbf{R} \equiv \frac{m_1\mathbf{r_1} + m_2\mathbf{r_2} + \ldots}{m_1 + m_2 + \ldots}$$

position vector **R** as the weighted average of the position vectors of the individual masses

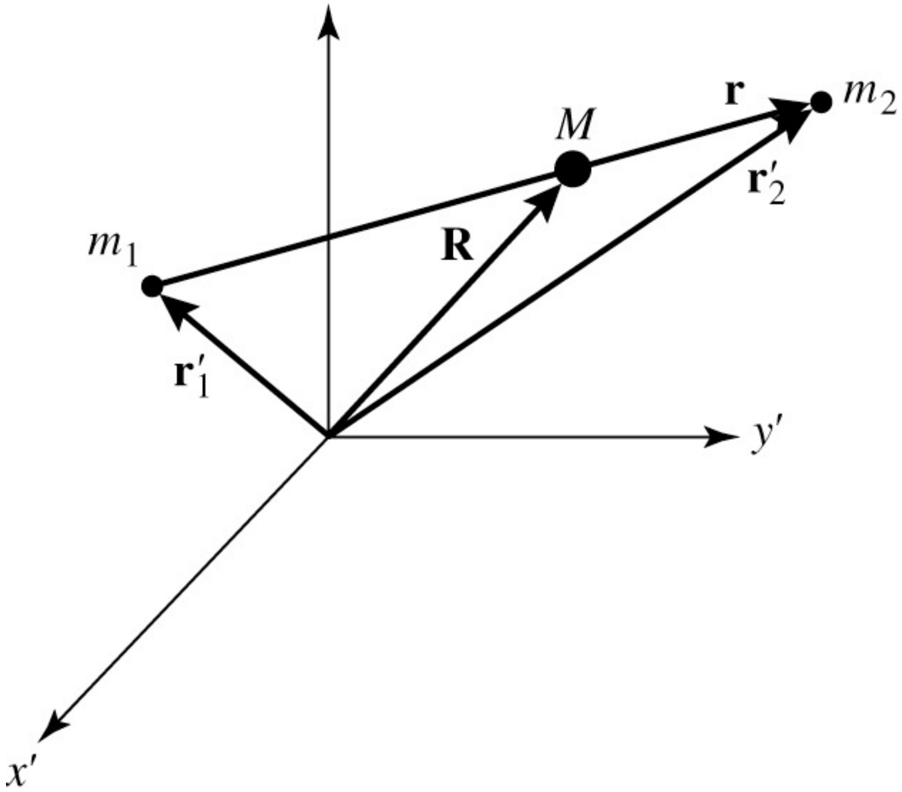
$$M\mathbf{R} = \sum_{i=1}^{n} m_i\mathbf{r_i}$$

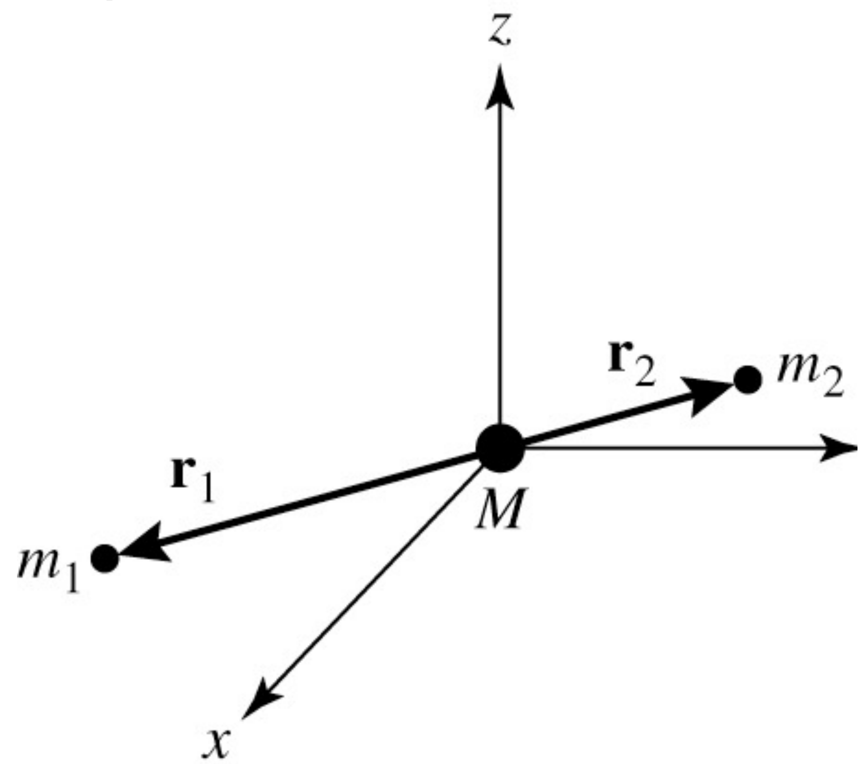
$$M\mathbf{V} = \sum_{i=1}^{n} m_i\mathbf{v_i}$$

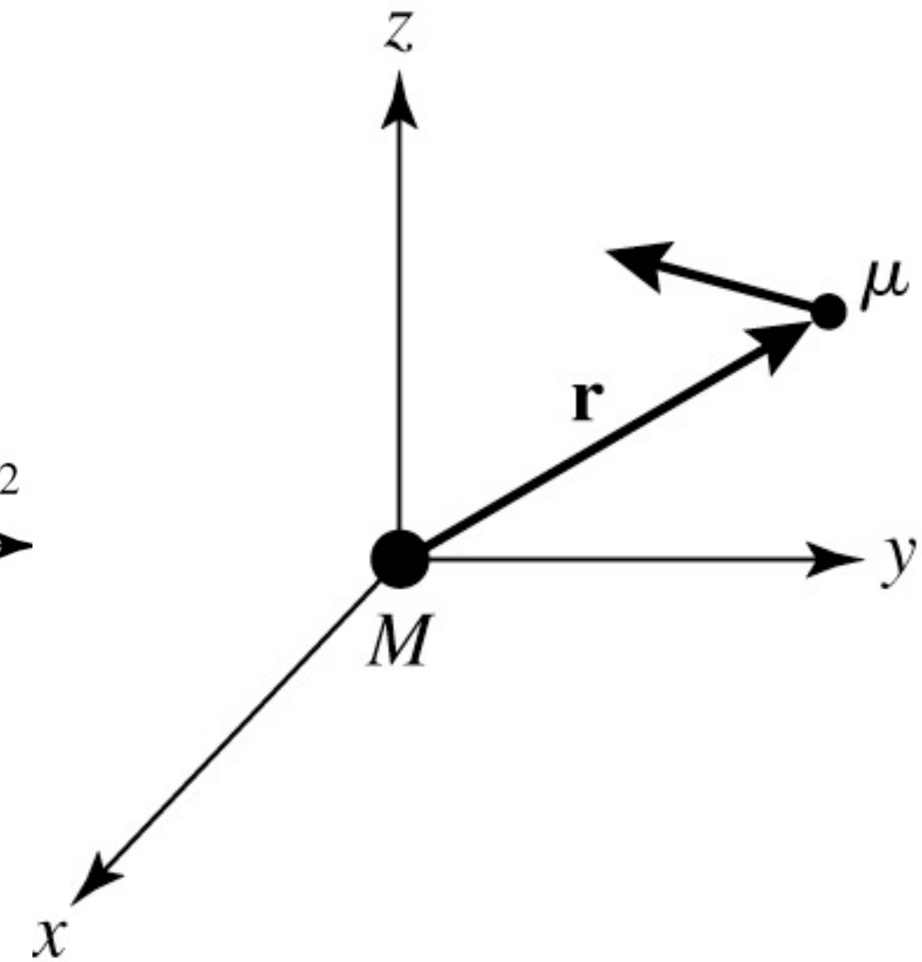
R is the position of the center of mass of the system, and **V** is the velocity of the center of mass. **P**=M**V** is the momentum of the center of mass.

Center of mass

$$\mathbf{r} = \mathbf{r}_2' - \mathbf{r}_1' \quad \text{displacement vector}$$







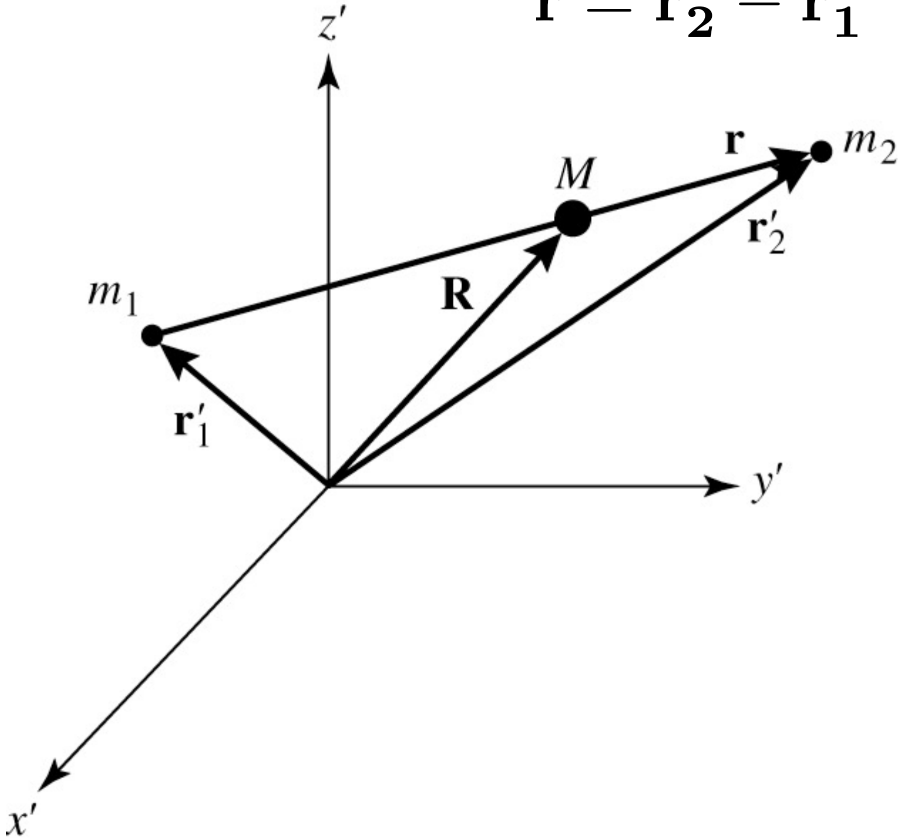
If all forces acting on individual particles in the system are due to other particles contained within the system (Newton's 3rd law):

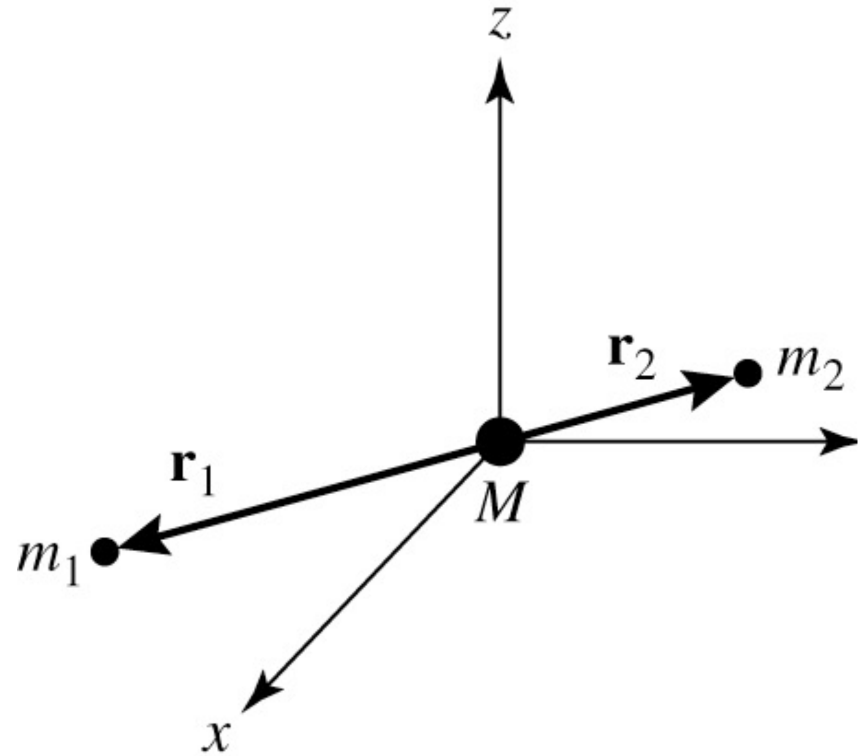
$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = 0$$

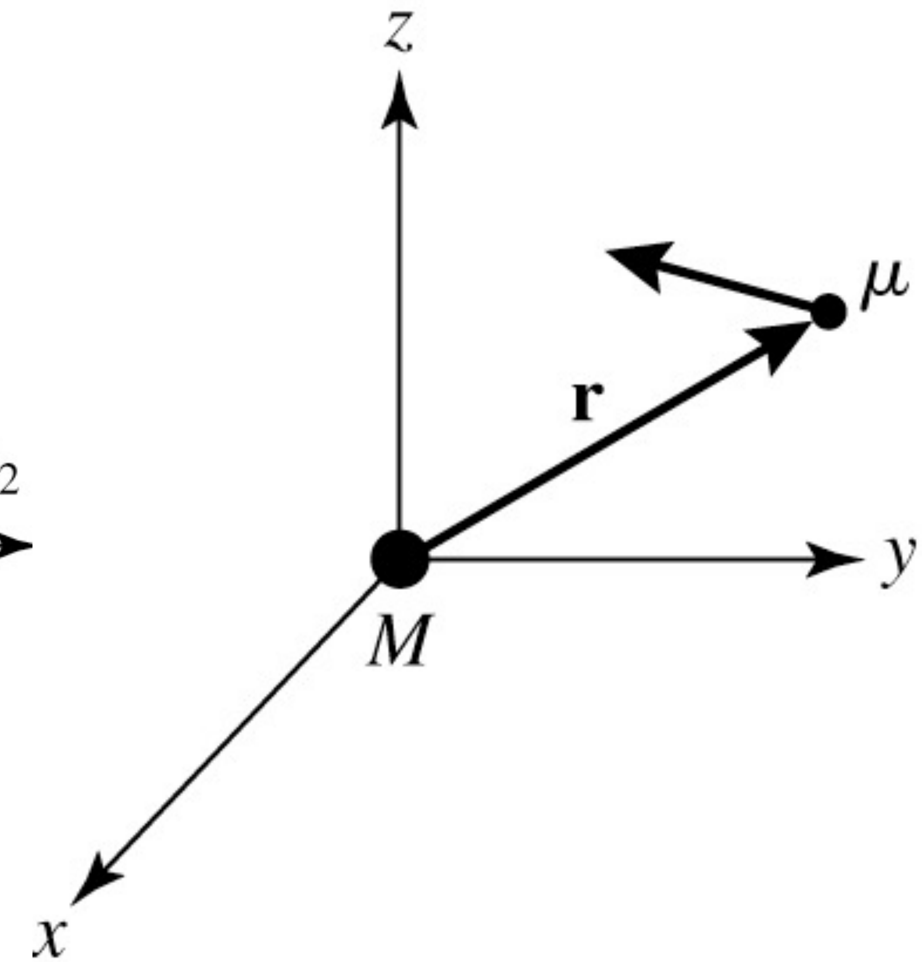
i.e., the center of mass does not accelerate if no external force exists, i.e., the reference frame associated to the center of mass is inertial

Center of mass

$\mathbf{r} = \mathbf{r}_2' - \mathbf{r}_1'$ displacement vector







It is therefore convenient to place the coordinate center at the center of mass, choosing **R**=0. Substituting $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{r}$, and defining the reduced mass (for a binary system) as:

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

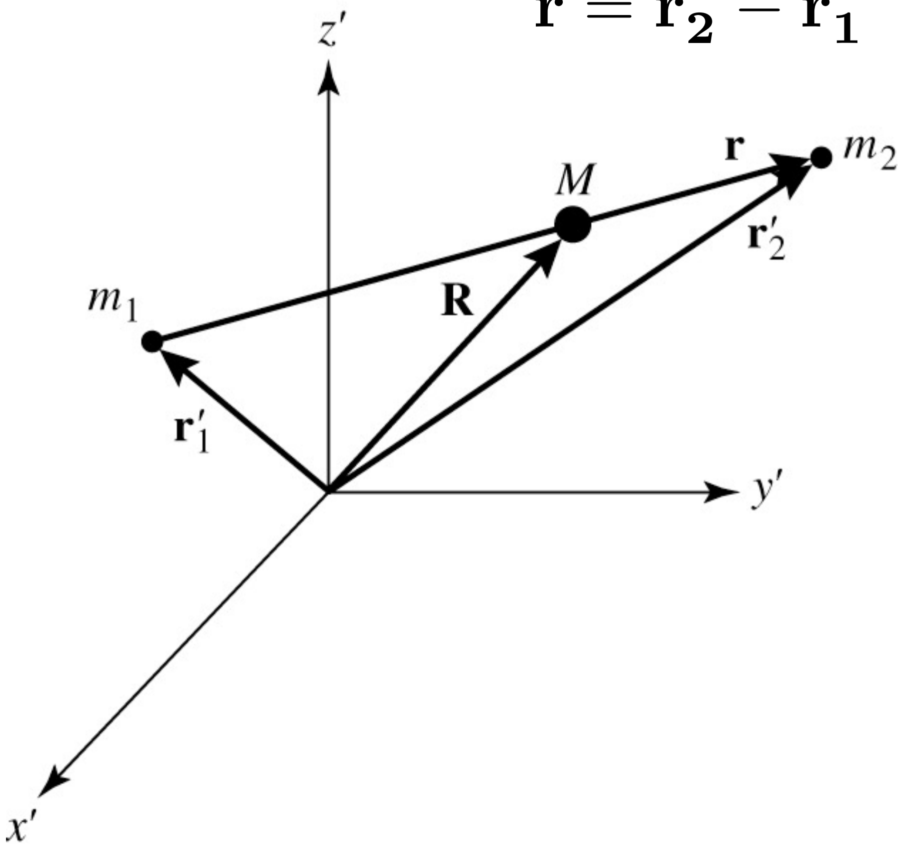
$$\Rightarrow \quad \mathbf{r}_1 = -\frac{\mu}{m_1}\mathbf{r} \qquad \mathbf{r}_2 = \frac{\mu}{m_2}\mathbf{r}$$

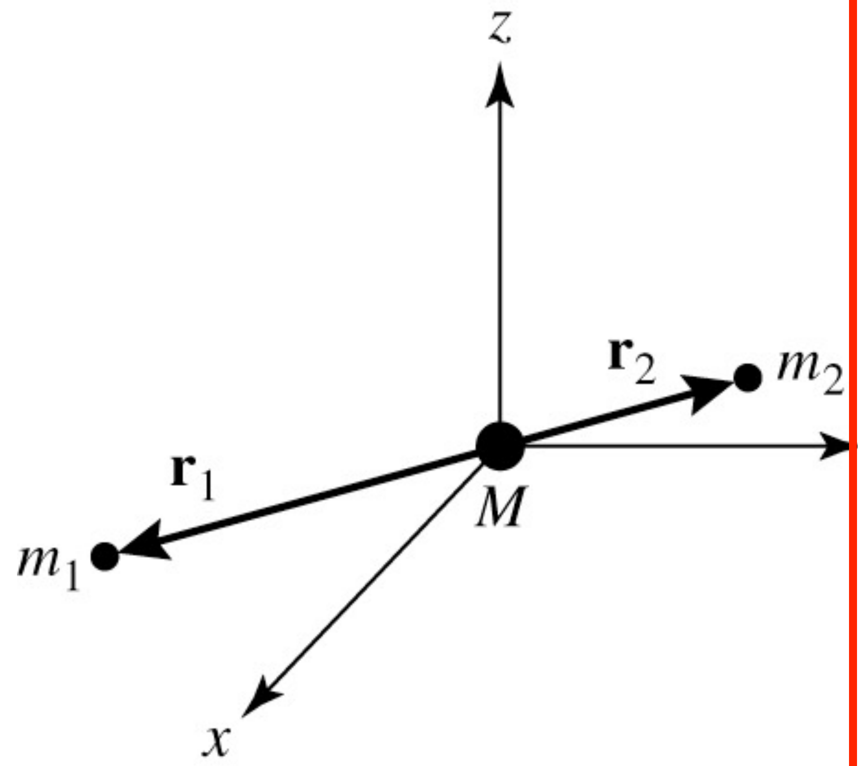
$$\Rightarrow \quad E = \frac{1}{2}\mu v^2 - G\frac{M\mu}{r}$$

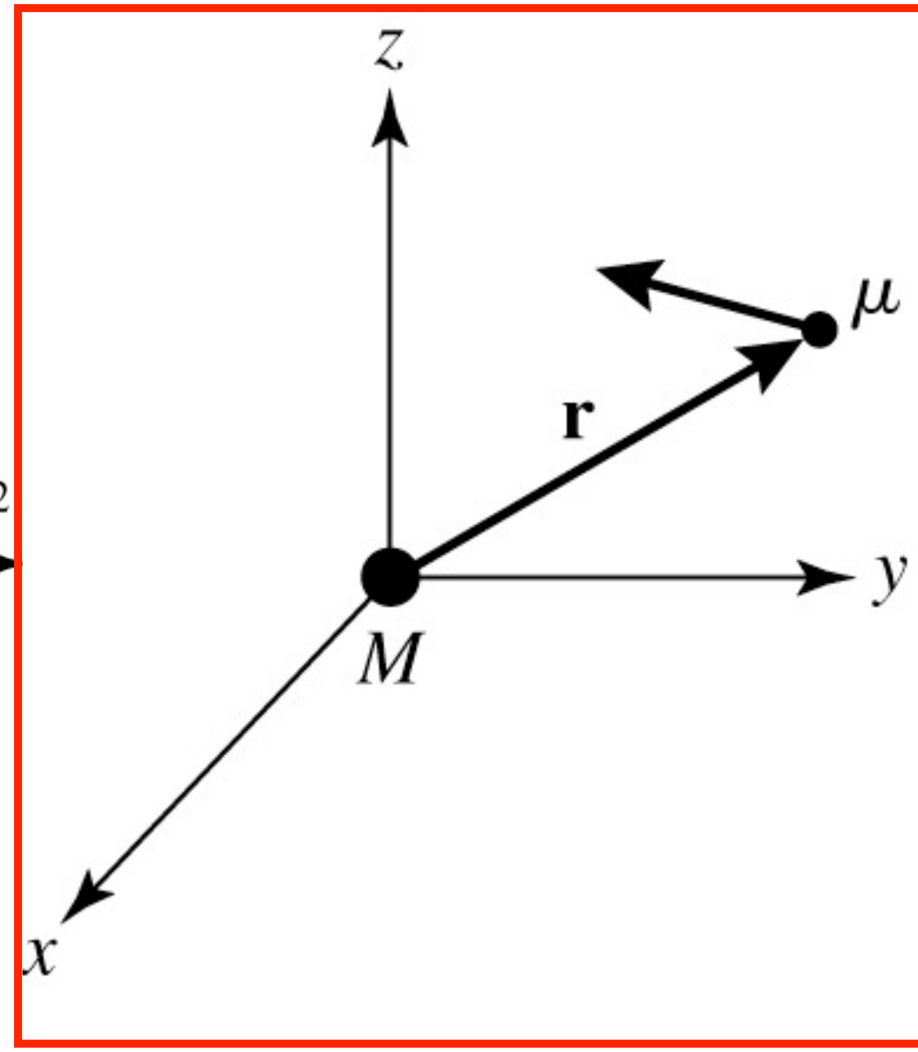
i.e., the total energy of the system is equal to the kinetic energy of the reduced mass plus the potential energy of the reduced mass moving about a mass M located and fixed at the origin.

Center of mass

$\mathbf{r} = \mathbf{r}'_2 - \mathbf{r}'_1$ displacement vector







Total angular momentum:

$$\mathbf{L} = m_1\mathbf{r_1} \times \mathbf{v_1} + m_2\mathbf{r_2} \times \mathbf{v_2} = \mu\mathbf{r} \times \mathbf{v} = \mathrm{r} \times \mathbf{p}$$

i.e., the total angular momentum equals the angular momentum of the reduced mass only

The two-body problem can be treated as an equivalent one-body problem with the reduced mass μ moving about a fixed mass M at a distance **r**.

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{F} = 0$$

L of a system is a constant for a central law force

Revisited Kepler's 1st Law

$$r = \frac{L^2/\mu^2}{GM(1 + e\cos\theta)} \quad \text{General equation of a conic section}$$

i.e., the path of the reduced mass about the center of mass under the influence of gravity is a conic section.

Elliptical orbits result from an attractive r^{-2} central-force law (i.e., gravity) when the total energy of the system is less than 0 ($E<0$, bound system); parabolic trajectories when $E=0$; and hyperbolic trajectories when $E>0$ (unbound systems).

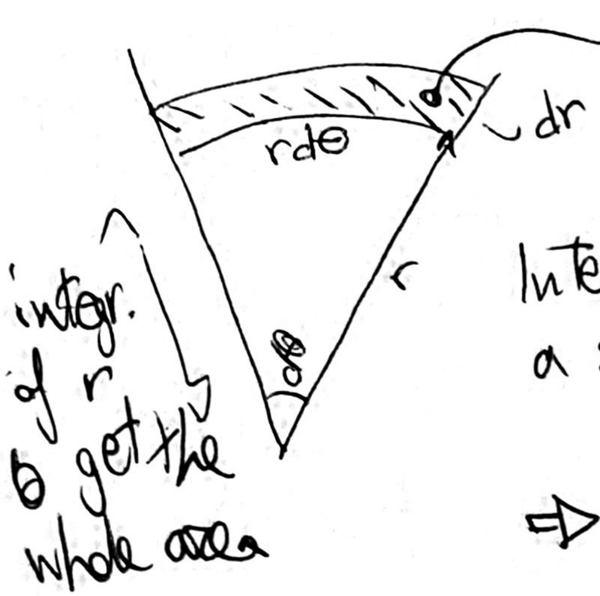
NOTE: both objects in a binary system move about the center of mass in ellipses, with the center of mass occupying one focus of each ellipse.

For closed planetary orbits, comparing the above with the previous equation of an ellipse, I obtain the total angular momentum of the system (max for e=0, i.e., circular motions):

$$L = \mu\sqrt{GMa(1-e^2)}$$

Derivation of Kepler's 2nd Law

DERIVATION of KEPLER's 2ND law



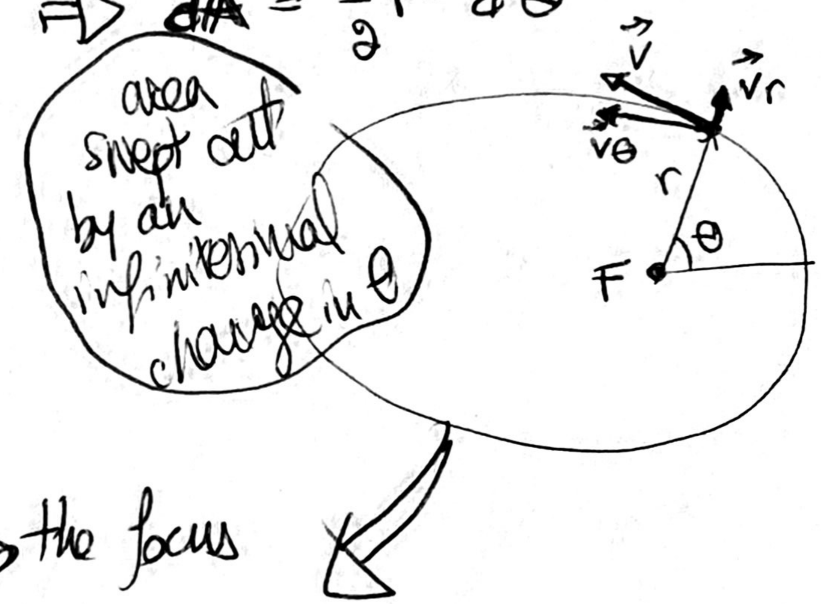
$d\sigma = r\,dr\,d\theta$

infinitesimal area element in polar coordinates

Integrating from the principal focus of the ellipse to a specific distance r $\Rightarrow dA = \frac{1}{2} r^2\, d\theta$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

time rate of change in area swept out by a line joining a point on the ellipse to the focus



Orbital velocity $\vec v = \vec v_r + \vec v_\theta = \frac{dr}{dt}\hat r + r\frac{d\theta}{dt}\hat\theta$

component along $\vec r$

component perpendicular to $\vec r$ w/ $v_\theta = r\frac{d\theta}{dt}$

substituting $v_\theta = r\frac{d\theta}{dt}$ $\rightarrow$ $\frac{dA}{dt} = \frac{1}{2} r v_\theta$

But $\vec r$ & $\vec v_\theta$ are perpendicular

$$r v_\theta = |\vec r \times \vec v| = \left|\frac{\vec L}{\mu}\right| = \frac{L}{\mu}$$

Derivation of Kepler's 2nd Law

$$\boxed{\frac{dA}{dt} = \frac{1}{2}\frac{L}{\mu}}$$ Kepler's 2ND law (revisited)

since L = constant, dA/dt is constant, ie: the time rate of change of the area swept out by a line connecting a planet to the focus of an ellipse is a constant, one-half of the orbital angular momentum per unit of mass.

Kepler's revised 1st law

From (II), perihelion $\theta = 0$ & aphelion ($\theta = \pi$) & using $L = \mu r v$ at these points, & using $r_p = a(1-e)$ & $r_a = a(1+e)$

$$v_p^2 = \frac{GM}{a}\left(\frac{1+e}{1-e}\right) \qquad v_a^2 = \frac{GM}{a}\left(\frac{1-e}{1+e}\right)$$

(9)

Total energy $E = \frac{1}{2}\mu v_p^2 - G\frac{M\mu}{r_p}$

w/ appropriate subs & re-arrangements

$$\boxed{E = -\frac{GM\mu}{2a} = -G\frac{m_1 m_2}{2a}}$$

ie: the total energy of a binary orbit depends only on the semi major axis a & is exactly one-half the time-averaged potential energy of the system

$$E = \frac{1}{2}\langle U \rangle .$$

From $\underbrace{-\frac{GM\mu}{2a}}_{E} = \underbrace{\frac{1}{2}\mu v^2 - G\frac{M\mu}{r}}_{K+U}$

& using $M = m_1 + m_2$

$$\boxed{v^2 = G(m_1 + m_2)\left(\frac{2}{r} - \frac{1}{a}\right)}$$

velocity of the reduced mass (or the relative velocity of m_1 & m_2)

Derivation of Kepler's 2nd Lav

$$\Rightarrow \quad \frac{dA}{dt} = \frac{1}{2}\frac{L}{\mu}$$

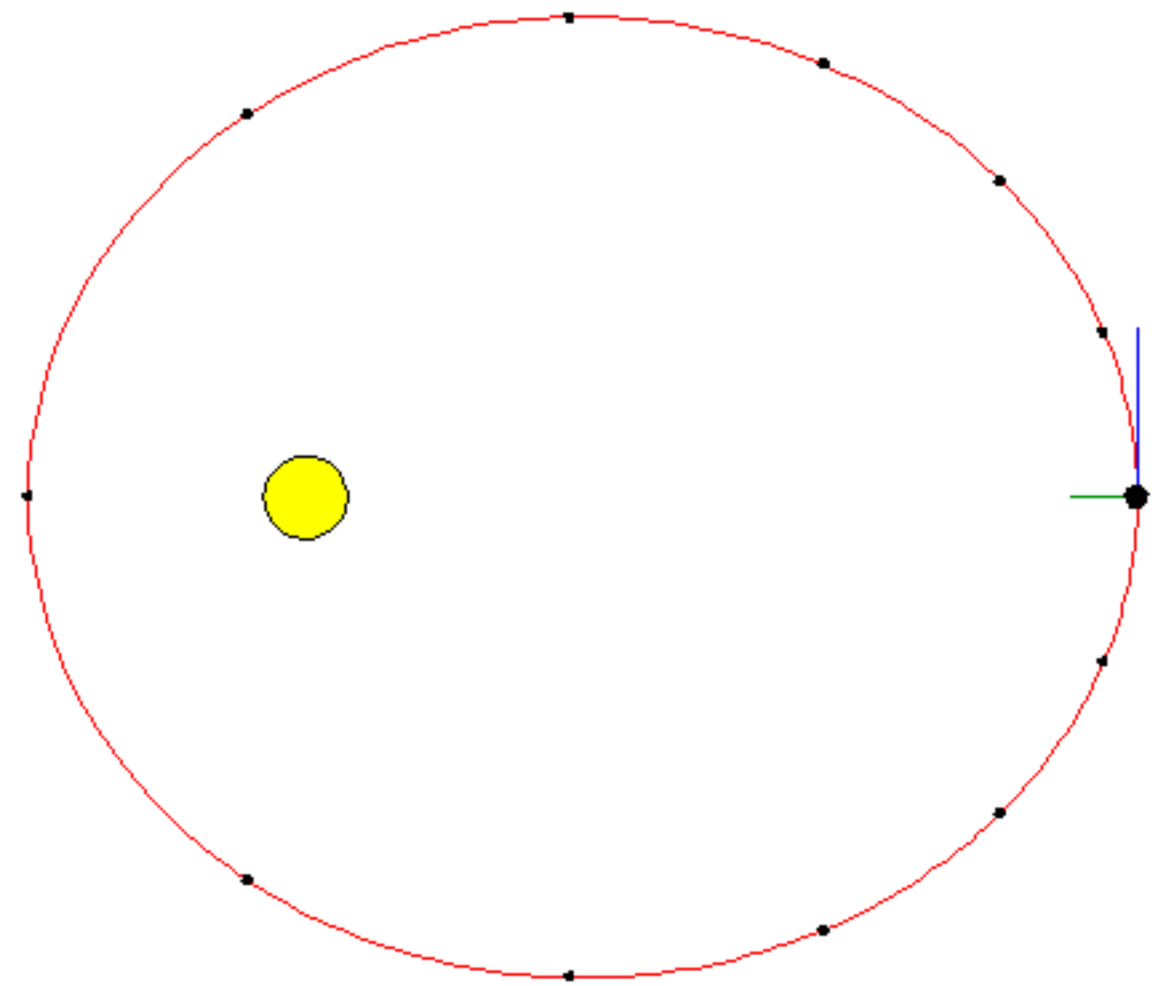
Kepler's (revis

since L = constant, dA/dt of change of the area swept planet to the focus of an of the orbital angular m

Kepler's revised 1st law

From (II), perihelion $\theta = 0$ & aphel at these points, & using $r_p = a(1$

$$\Rightarrow \quad v_p^2 = \frac{GM}{a}\left(\frac{1+e}{1-e}\right) \qquad v_a^2 =$$



(9)

Total energy $E = \frac{1}{2}\mu v_p^2 - G\frac{M\mu}{r_p}$

w/ appropriate subs & re-arrangements

$$\Rightarrow \quad E = -\frac{GM\mu}{2a} = -G\frac{m_1 m_2}{2a}$$

ie: the total energy of a binary orbit depends only on the semi major axis a & is exactly one-half the time-averaged potential energy of the system

$$E = \frac{1}{2}\langle U \rangle.$$

From $\underbrace{-\frac{GM\mu}{2a}}_{E} = \underbrace{\frac{1}{2}\mu v^2 - G\frac{M\mu}{r}}_{K+U}$

& using $M = m_1 + m_2$

$$\Rightarrow \quad v^2 = G(m_1 + m_2)\left(\frac{2}{r} - \frac{1}{a}\right)$$

velocity of the reduced mass (or the relative velocity of m_1 & m_2

Derivation of Kepler's 2nd Law

$$\Rightarrow \boxed{\frac{dA}{dt} = \frac{1}{2}\frac{L}{\mu}}$$ Kepler's 2ND law (revisited

since L = constant, dA/dt is constant, ie: the time rate of change of the area swept out by a line connecting a planet to the focus of an ellipse is a constant, one-half of the orbital angular momentum per unit of mass.

Kepler's revised 1st law

From (II), perihelion $\theta = 0$ & aphelion ($\theta = \pi$) & using $L = \mu r v$ at these points, & using $r_p = a(1-e)$ & $r_a = a(1+e)$

$$\Rightarrow v_p^2 = \frac{GM}{a}\left(\frac{1+e}{1-e}\right) \qquad v_a^2 = \frac{GM}{a}\left(\frac{1-e}{1+e}\right)$$

Total energy $E = \frac{1}{2}\mu v_p^2 - G\frac{M\mu}{r_p}$ ⑨

w/ appropriate subs & re-arrangements

$$\Rightarrow \boxed{E = -\frac{GM\mu}{2a} = -G\frac{m_1 m_2}{2a}}$$

ie: the total energy of a binary orbit depends only on the semi major axis a & is exactly one-half the time-averaged potential energy of the system $E = \frac{1}{2}\langle U \rangle$.

From
$$\overbrace{-\frac{GM\mu}{2a}}^{E} = \overbrace{\frac{1}{2}\mu v^2 - G\frac{M\mu}{r}}^{K+U}$$

& using $M = m_1 + m_2$

$$\Rightarrow \boxed{v^2 = G(m_1 + m_2)\left(\frac{2}{r} - \frac{1}{a}\right)}$$

velocity of the reduced mass (or the relative velocity of m_1 & m_2

Derivation of Kepler's 3rd Law

DERIVATION OF KEPLER'S 3RD LAW

From $\frac{dA}{dt} = \frac{1}{2}\frac{L}{\mu}$ & integrating over one orbital period P

$\Rightarrow A = \frac{1}{2}\frac{L}{\mu}P$. Using $A = \pi ab$

$$\Rightarrow P^2 = \frac{4\pi^2 a^2 b^2 \mu^2}{L^2}$$

Using $b^2 = a^2(1-e^2)$ & $L = \mu\sqrt{GMa(1-e^2)}$

$$\Rightarrow \boxed{P^2 = \frac{4\pi^2}{G(m_1+m_2)}a^3}$$ Kepler's 3rd law revisited (Δ)

For the solar system $m_1+m_2 = M_\odot + m_{planet} \simeq M_\odot$

(Δ) is the most direct way of obtaining masses of celestial objects, a critical parameter in understanding a wide range of phenomena.

Knowledge of P & a $\rightarrow$ $M = m_1+m_2$ can be measured.

Individual masses can be derived, if relative distances to the center of mass are known.