Energy Transportation

Reading assignment

THURSDAY 11/3: 12.2 (3 pages)+13.1+13.2

Homework Assignment #4 due by:

TUESDAY 11/10 before 9AM.

Note that TUESDAY is a WED schedule —> no class

MIDTERM EXAM: THURSDAY Nov. 12

Eddington Luminosity

opacity (coefficient of absorption) [m²/kg]
cross section of absorbing photons per unit of mass

$$\frac{dP_{rad}}{dr} = -\frac{\tilde{\kappa}\rho}{c} F_{rad}$$

pressure gradient

radiative flux

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$$F = \frac{L}{4\pi r^2}$$

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If $L > L_{Edd}$, mass loss occurs, driven by radiative pressure.

For massive stars (T~50,000K), H is completely ionized in the atmosphere, hence the opacity is due to electron scattering.

X=0.7

$$\bar{k}_{e^- \ scattering} = 0.02(1 + X) \ \ \mathrm{m}^2 \ \mathrm{kg}^{-1}$$

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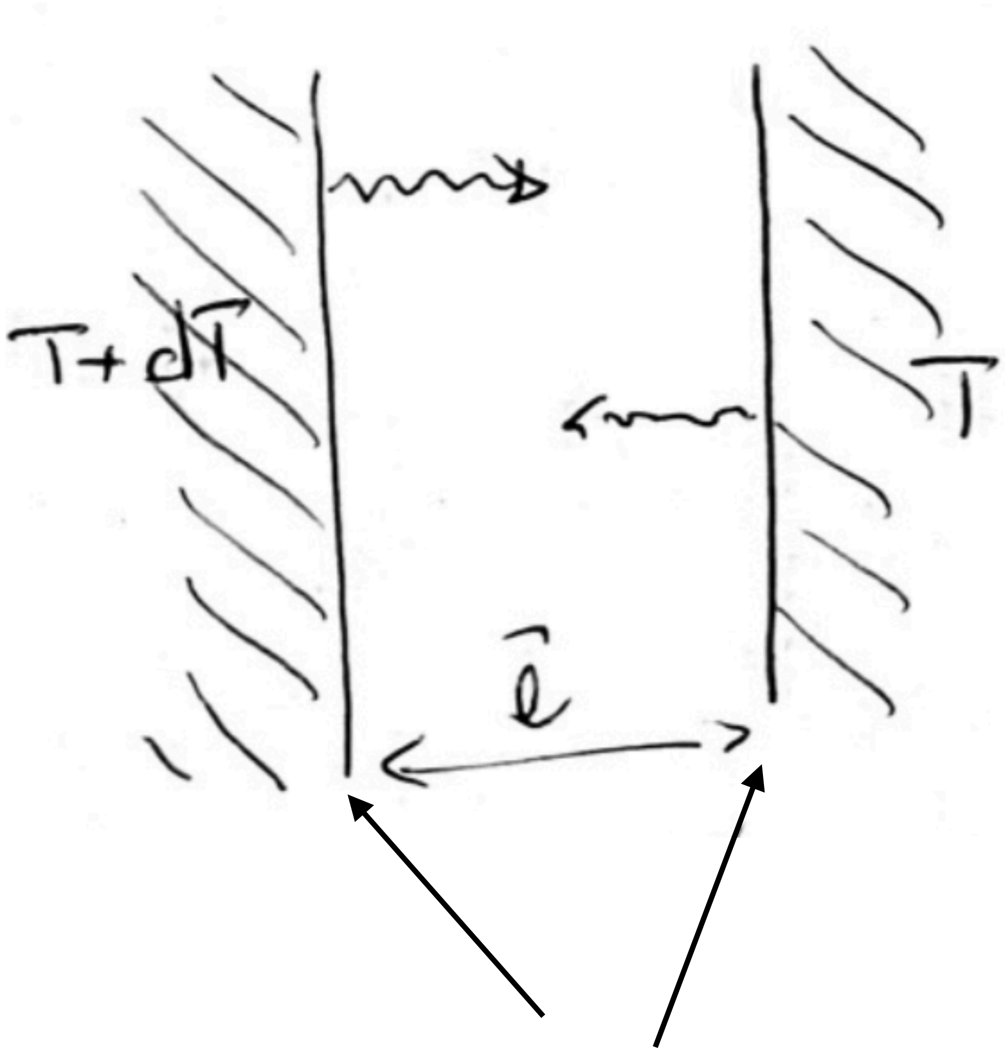
For M=90 M_{Sun}, $L_{Edd} = 3.4 \times 10^6 \ L_{Sun}$, i.e., 3x the expected main-sequence value.
The envelopes of very massive main-sequence stars are loosely bound at best, and suffer from large amount of mass loss.

Energy Transportation

There are three different energy transport mechanisms in the stellar interior:

1. **Radiation** (photons are absorbed and re-emitted in nearly random directions as they encounter matter)
2. **Convection** (can be very efficient in many regions of a star, with hot, buoyant mass elements carrying excess energy outward, while cool elements fall inward)
3. **Conduction** (transport of heat via collisions between particles)

Radiative transfer



Semi-infinite blackbody planes (they absorb all radiation)

Net difference in the exiting flux over the entering flux per unit of area:

$$H = \sigma\left[(T+dT)^4 - T^4\right] \simeq 4\sigma T^3 dT$$

H is the heat (energy) flow in a vacuum between two parallel blackbodies of temperature difference dT

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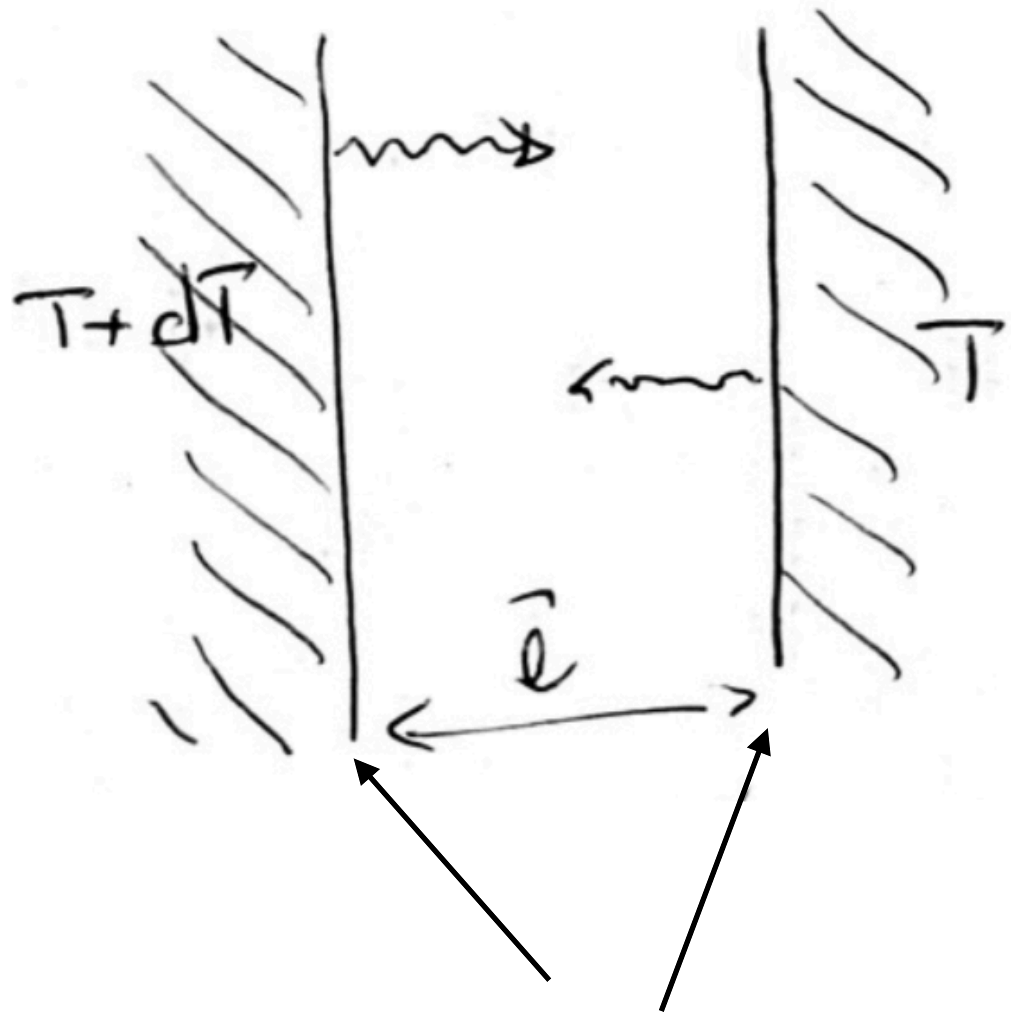

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$$\boxed{H = -\frac{4ac}{3\tilde{\kappa}\rho} T^3 \frac{dT}{dr}}$$

Rosseland mean opacity [m²/kg], i.e., the cross section of absorption per unit of mass

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eq.

$$\frac{dT}{dr} = -\frac{3\tilde{\kappa}\rho L(r)}{4\pi r^2 \, 4acT^3}$$

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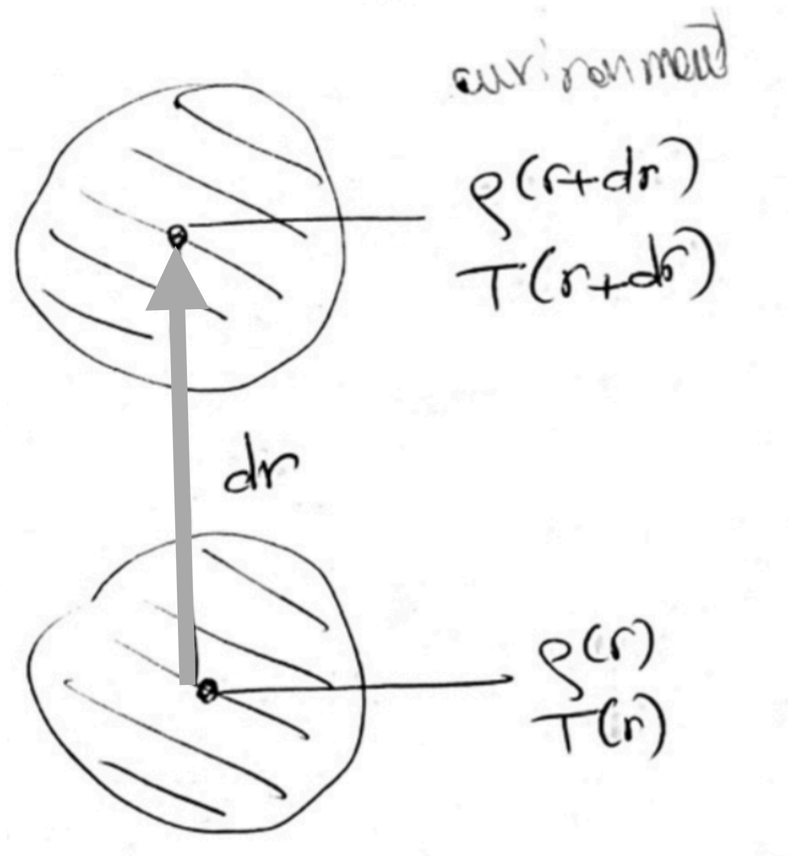
In a static model, the heat flux must be sufficiently large to carry out all the energy liberated within a given sphere, and this requirement establishes the temperature gradient dT/dr. **IF dT/dr is too large, the instability to convective gas motion.**

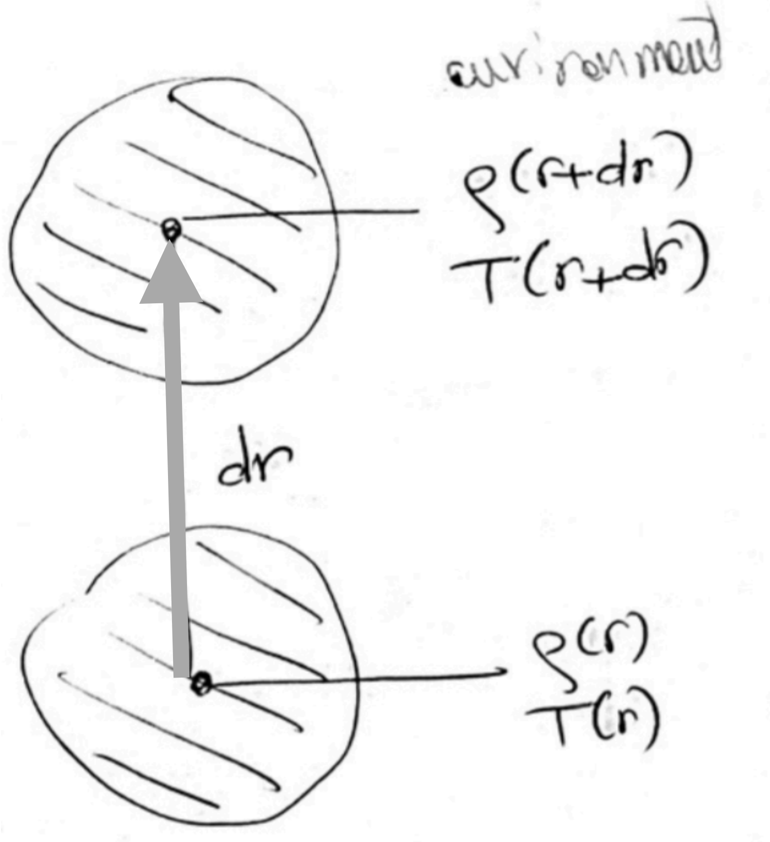
Convective instability



The mass element is displaced by dr without exchanging heat with the environment (i.e., adiabatic change).

If expanded density is larger than the density of the environment, then the displaced element is denser than the environment, and will settle back under gravity, otherwise net buoyant force, and it will continue upward.

Convective instability



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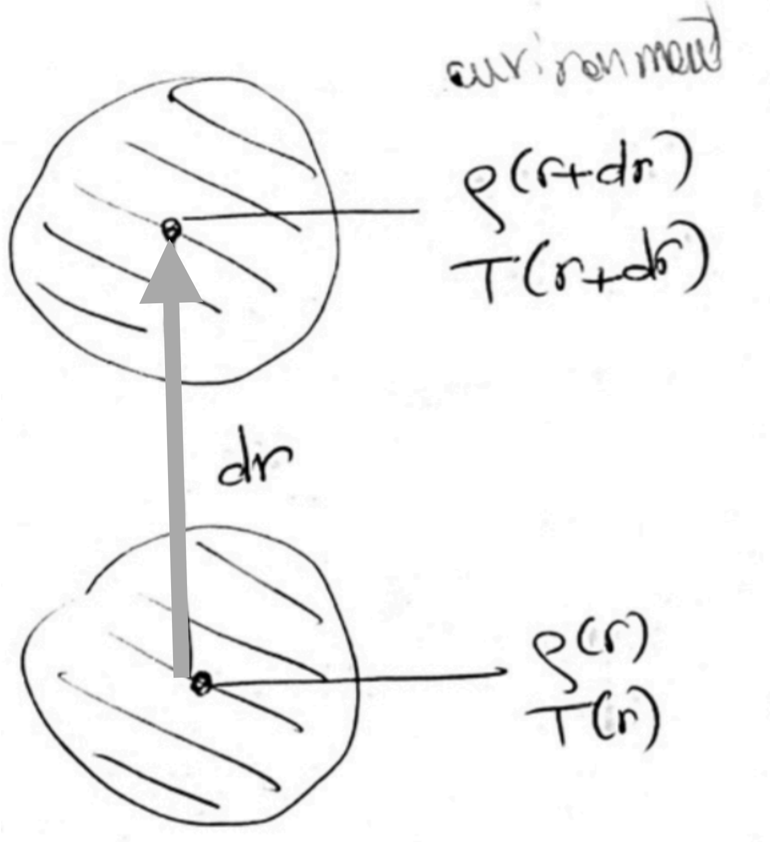

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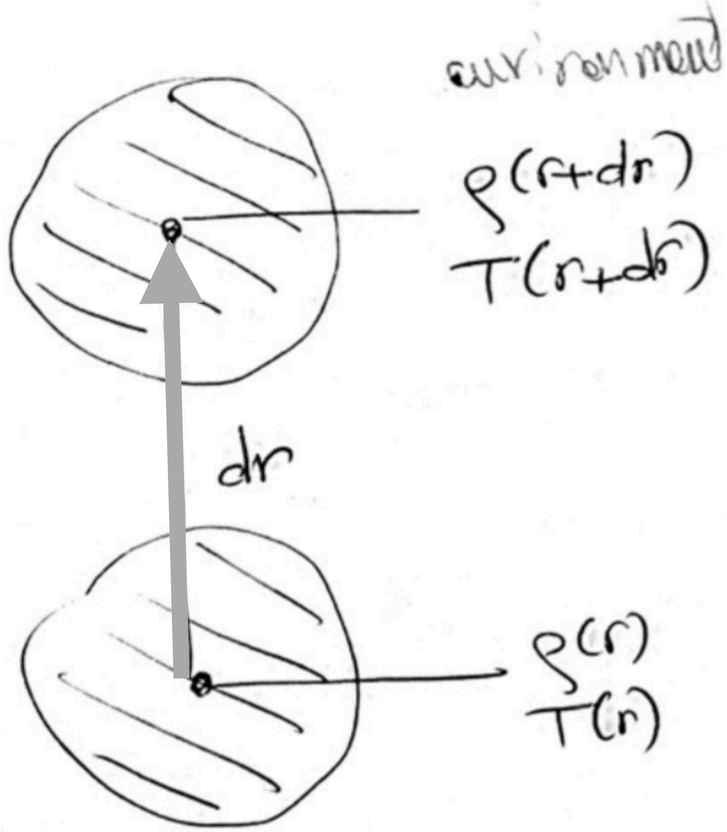
Ideal gas law

$$\frac{dP}{dr} = -\frac{P}{\mu}\frac{d\mu}{dr} + \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

differentiating

Convective instability



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differentiating

For adiabatic expansions $P = K\rho^{\gamma}$

$$\hookrightarrow \frac{dP}{dr} = \gamma \frac{P}{\rho}\frac{d\rho}{dr}$$

Assuming $\mu = \text{const} \longrightarrow \frac{d\mu}{dr} = 0$

$$\Rightarrow \frac{dP}{dr} = \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

$$\frac{dP}{dr} = \gamma \frac{P}{\rho}\frac{d\rho}{dr}$$

Combining these two

$$\Rightarrow \frac{dT}{dr} = \frac{T}{\rho}\frac{d\rho}{dr}(\gamma - 1)$$

using $\frac{dP}{dr} = \gamma \frac{P}{\rho}\frac{d\rho}{dr}$

ADIABATIC TEMPERATURE GRADIENT

$$\Rightarrow \left.\frac{dT}{dr}\right|_{ADIAB} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

Using $P = \frac{\rho K T}{\mu m_H}$ & $\frac{dP}{dr} = -\frac{G M_r \rho}{r^2}$

$$\Rightarrow \boxed{\left.\frac{dT}{dr}\right|_{ADIAB} = -\left(1 - \frac{1}{\gamma}\right) \frac{\mu m_H}{K} \frac{G M_r}{r^2}}$$

5B) - basic equation of stellar structure when unstable against convective motions.

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NOTE: If $\left|\left(\frac{dT}{dr}\right)_{star}\right| < \left|\left(\frac{dT}{dr}\right)_{adiab.}\right| \Rightarrow$ **STABILITY CONDITION (no convection)**

$$\Leftrightarrow \boxed{\underbrace{\left(\frac{dT}{dr}\right)_{star}}_{5A} > \underbrace{\left.\frac{dT}{dr}\right|_{ADIAB}}_{5B}}$$

i.e., if the temperature changes too rapidly with distance, instability against convection exists

Using $\frac{dT}{dr} = -\frac{3\tilde{K}\rho\, L(r)}{4\pi r^2\, 4acT^3}$ (5A)

$$\Rightarrow \quad -\frac{3}{4ac}\,\frac{\tilde{K}\rho}{T^3}\,\frac{L(r)}{4\pi r^2} > -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_H}{k}\,\frac{G M_r}{r^2}$$

Using $P = \frac{\rho k T}{\mu m_H} \rightarrow \frac{\mu m_H}{\rho k} = \frac{T}{P}$

$$\boxed{L(r) \leq \frac{16\pi a c G}{3\tilde{K}}\left(1-\frac{1}{\gamma}\right)\frac{T^4}{P} M_r}$$

Using $\frac{dT}{dr} = -\frac{3\tilde{\kappa}\rho L(r)}{4\pi r^2 \, 4acT^3}$ (5A)

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i.e., if the luminosity required to maintain energy balance is larger than this value, the energy will have to be carried by convection

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For an ideal non-degenerate gas, $\gamma = \frac{5}{3}$

$\hookrightarrow L(r) \leq 1.22 \times 10^{-18} \frac{\mu T^3}{\tilde{\kappa}\rho} M(r) \frac{erg}{s}$

In the ionization zone, the opacity is very large, hence the luminosity there exceeds the upper limit for radiative equilibrium, and convection zone develops.

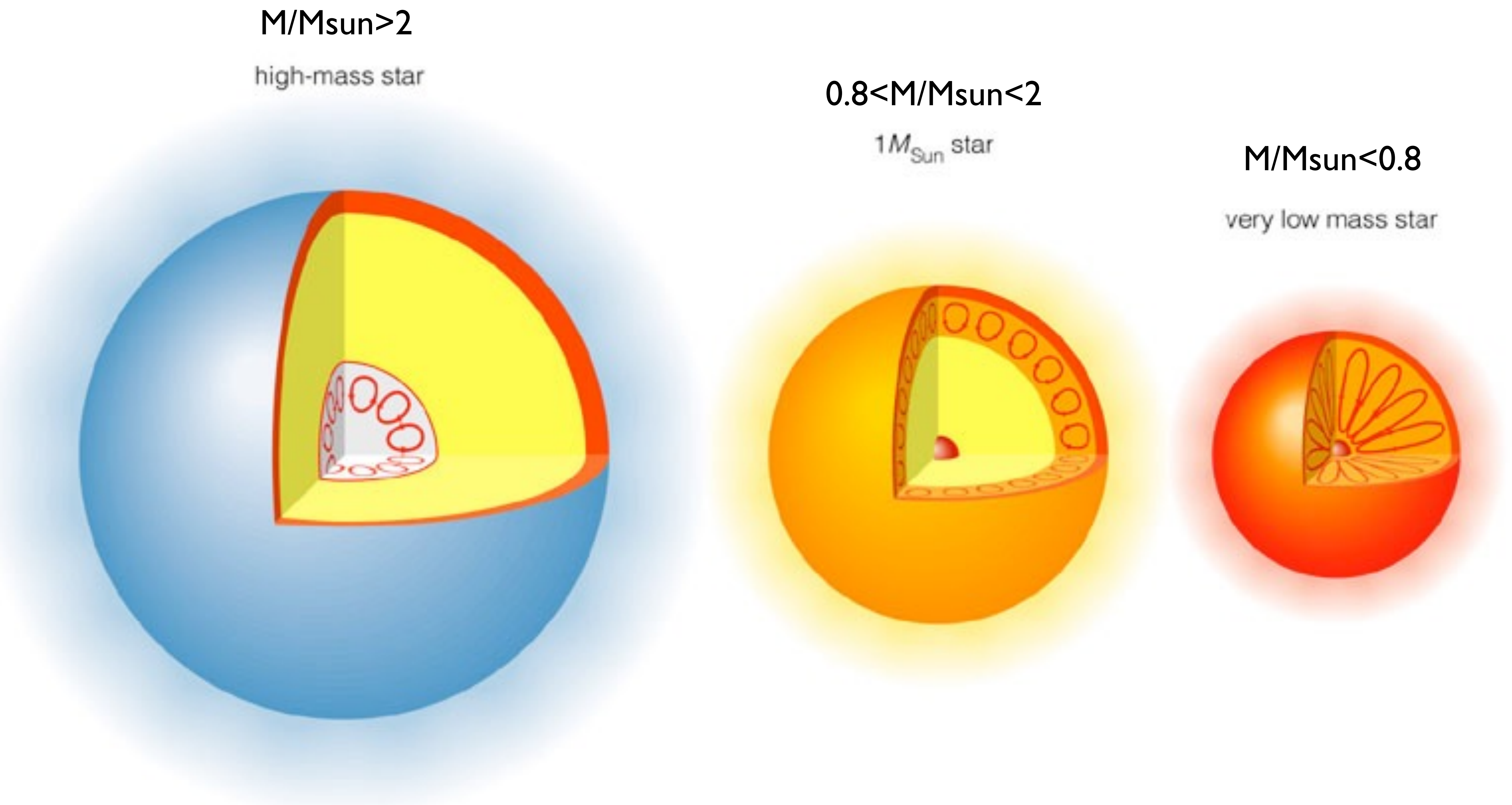
IF medium is unstable —>

an adiabatically rising element is less dense and hotter than the environment —>

since the rising mass is hotter than the environment, heat will leave from it to the surroundings —>

the net effect is the transport of heat to material at a lower temperature —>

energy transport (heating the outer regions and cooling the inner regions.


M/Msun>2
high-mass star
0.8<M/Msun<2
$1M_{Sun}$ star
M/Msun<0.8
very low mass star

Summary Equations of Stellar Structure

① CONSERVATION OF MASS $\frac{dM_r}{dr} = 4\pi r^2 \rho$

② EQ. OF HYDROSTATIC EQUILIBRIUM $\frac{dP}{dr} = -\rho \frac{GM_r}{r^2}$ $\neq 0$ when star is contracting slowly

④ EQ. OF ENERGY BALANCE $\frac{dL(r)}{dr} = 4\pi \rho r^2 \left[\epsilon(r) - T\frac{dS}{dt}\right]$

$\epsilon = \epsilon_{NUCLEAR} + \epsilon_{GRAVITY}$

IF the structure of the star is changing over time

w/ $L(r)$ energy per second emitted by the sphere of radius r

S entropy per unit of mass

Power generated by gram of matter by nuclear reactions $\epsilon(r)$

<u>NOTE</u>: $\epsilon_\nu = -\frac{1}{4\pi r^2 \rho}\frac{dL_\nu(r)}{dr}$

ie: $\epsilon = \epsilon_n - \epsilon_\nu$

ie: the neutrino power produced per gram is regarded as a negative contribution to ϵ, ie. $L(r)$ does not include the neutrino power.

⑤ $\frac{dT}{dr} = -\frac{3}{4ac}\frac{\tilde{\kappa}\rho}{T^3}\frac{L(r)}{4\pi r^2}$ Ⓐ If material in radiative equilibrium

$\frac{dT}{dr} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2}$ Ⓑ If convectively unstable —> adiabatic convection

k: Boltzmann constant

5B is used instead of 5A if $\left.\frac{dT}{dr}\right|_{\text{From 5A}} > \left.\frac{dT}{dr}\right|_{\text{From 5B}}$

<u>NOTE</u>: 5B is often inadequate near the stellar surface

CONSTITUTIVE RELATIONS

\+ EQ. OF STATE ③ $\rho = \rho(P, T, \mu)$

POWER PER GRAM liberated by nuclear reactions $\epsilon = \epsilon(\rho, T, \mu)$

OPACITY $\tilde{\kappa} = \bar{\kappa}(\rho, T, \mu)$

⇒ 4 variables $[P(r), M(r), T(r), L(r)]$ w/

4 differential equations & 4 boundary conditions

$$\begin{cases} L_c = 0 \\ M_c = 0 \\ P(r=R) = 0 \\ T(r=R) = 0 \end{cases}$$

SOLUTION

M(r), L(r), T(r), P(r) o mass density (r), from the center of the star to its surface

NOTE: If the star is not expanding / contracting slowly,

$\Rightarrow$ (4) $\dfrac{dL(r)}{dr} = 4\pi r^2 \rho\, \epsilon(r)$, ie: I have

4 ORDINARY DIFFERENTIAL EQUATIONS (ODEs)

If instead there is slow contraction / expansion

$\Rightarrow$ (4') $\dfrac{dL(r)}{dr} = 4\pi r^2 \rho \left(\epsilon(r) - T \dfrac{dS}{dt} \right)$

PARTIAL DIFFERENTIAL EQ. (PDEs)

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PARTIAL DIFFERENTIAL EQ. (PDEs)

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$\boxed{2}$ when changes in the structure of the star are rapid enough that acceleration cannot be neglected

$\Rightarrow$ (2') $\rho \dfrac{d^2 r}{dt^2} = -G \dfrac{M_r \rho}{r^2} - \dfrac{dP}{dr}$

This is the case for supernova explosions or during stellar pulsations (free fall). Unless in these cases, $\rho \dfrac{d^2 r}{dt^2} = 0 \Rightarrow$ EQ. of HYDROSTATIC EQUILIBRIUM

(4) The term $-4\pi \rho r^2 T \frac{dS}{dt}$ is instead essential to calculations of stellar evolution, & can be ignored ONLY in the long-lived static phases of stellar evolution. HOWEVER, when a star contracts or expands, even at a 'modest' rate, the term $\frac{dS}{dt}$ becomes important. Moreover, this means that the structure of the star cannot be computed without knowledge of the star's previous history.

IF $T\frac{dS}{dt}$ small $\longrightarrow$ approximation $T \frac{S(t_0) - S(t_0-\Delta t)}{\Delta t}$

↳ numerical solution

$T\frac{dS}{dt}$ large $\longrightarrow$ Δt has to be decreased to stay in the above approximation $\longrightarrow$ more computationally expensive.

For a given stellar mass and chemical composition, a sequence of models describing the star as it ages is computed. This is repeated for a spectrum of masses and compositions.