Stellar Structure

Reading assignment

THURSDAY 10/20: Chapter 10.3

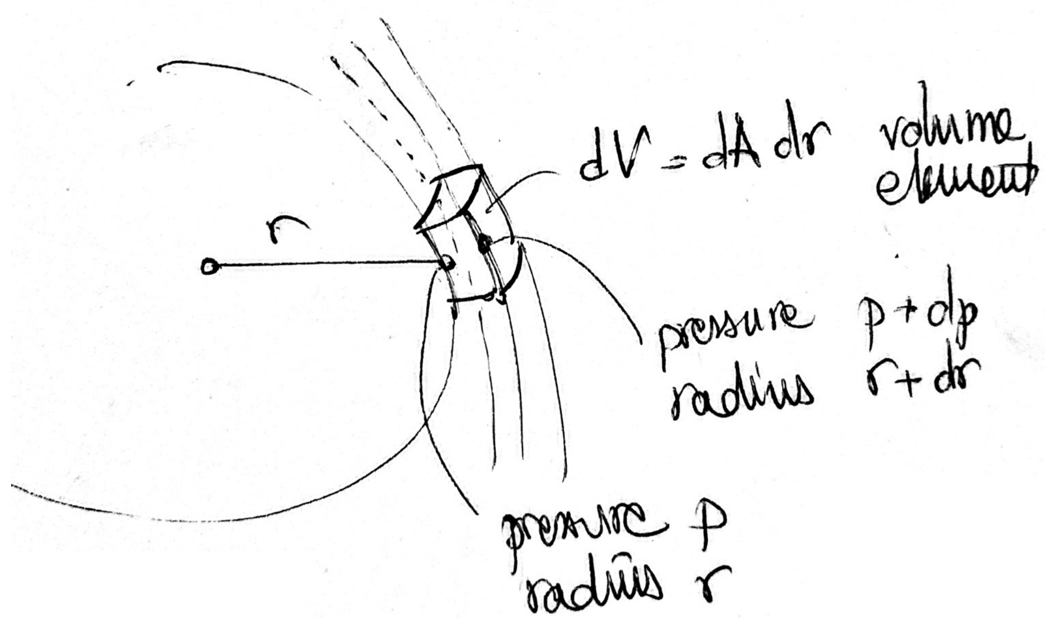
Homework Assignment #3 due by:

TUESDAY 10/27 before beginning of class

Stellar Structure

Consider the problem of an isolated, non-rotating, non-magnetic, spherical mass of gas held together by gravity (BUT stars do rotate, frequently occur in binary pairs, and have observable magnetic fields —> these are treated as perturbations of model).

Sphere of gas in hydrostatic equilibrium (mechanic equilibrium)



$$M_r = \int_0^{r'} 4\pi r^2 \rho(r)\, dr$$

1) eq. of conservation of mass

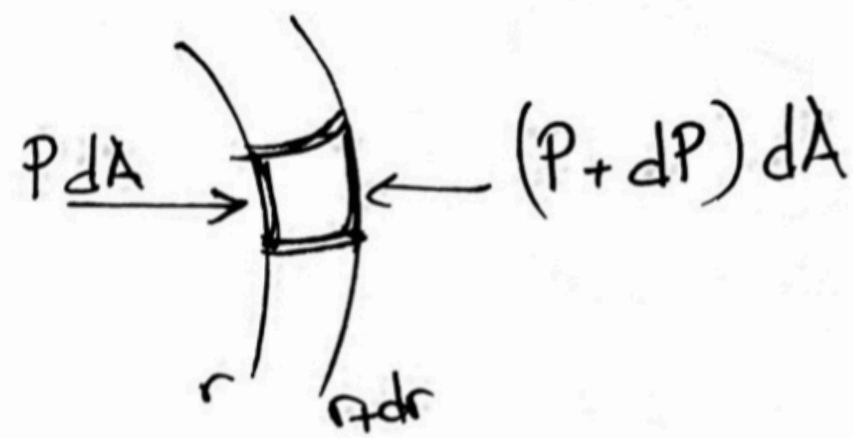
$$\underbrace{\rho \ddot{r}}_{\sim ma} = \underbrace{F_g + F_{P_r} + F_{P_{r+dr}}}_{\text{force}}$$

mass · acceleration

$$\ddot{r} = \frac{d^2 r}{dt^2}$$

w/ $F_g < 0$ gravitational force (inward) $\quad F_g = -\frac{G M_r \rho}{r^2}$

F_{P_r}, $F_{P_{r+dr}}$ pressure forces from above & below



$$\Rightarrow \quad F_P = \frac{P\,dA - (P+dP)\,dA}{dV} = -\frac{dP}{dr}$$

Effective pressure force on the volume element

$$\Rightarrow \rho \ddot{r} = -\frac{GM_r}{r^2}\rho - \frac{dP}{dr}$$

Equation of motion

Acceleration due to gravity

Pressure gradient (balancing the gravitational pull)

IF HYDROSTATIC EQUILIBRIUM: $\rho \ddot{r} = 0$

$$\boxed{\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho}$$

2) eq. of hydrostatic equilibrium (conservation of momentum)

NOTE: in order for a star to be static, a **pressure gradient** dP/dr must exist from the center to the surface to counteract the force of gravity. It is the pressure gradient that supports the star (NOT the pressure itself). The pressure must decrease with increasing radius (larger in the interior than near the surface).

SINCE $M_r \neq 0$ EXCEPT @ $r=0$ $\Rightarrow$ $\frac{dP}{dr}$ ALWAYS FINITE OR NULL, & IT IS CONTINUOUS

$\left.\frac{dP}{dr}\right|_c = 0$ SINCE $\left.M_r\right|_c = 0$

NOTE: Properties of central temperature and pressure in the Sun, T_C and P_C.

$$M_r = 1\, M_\odot \qquad \rho = \bar{\rho}_\odot = \frac{M}{\frac{4}{3}\pi R_\odot^3} = 1410\ \frac{kg}{m^3}$$
$$r = 1\, R_\odot$$

$$\frac{dP}{dr} \approx \frac{P_{surface} - P_c}{R_{surface} - 0} = -\frac{P_c}{R_\odot} = -\frac{G M_r \rho}{r^2} = -\frac{G M_\odot \bar{\rho}_\odot}{R_\odot^2}$$

$P_{surface} = 0$

From Eq. of Hydrostatic equilibrium

$$\Rightarrow P_c = \frac{G M_\odot \bar{\rho}_\odot}{R_\odot} = 2.7 \times 10^{14}\ N/m^2$$
$$\simeq 2.7 \times 10^{15}\ dyne/cm^2$$
$$\simeq 2.7 \times 10^{15}\ barye$$
$$\simeq 2.7 \times 10^{9}\ atm$$

ENORMOUS CENTRAL PRESSURE

$$P_c = \int_{P_s}^{P_c} dP = -\int_{R_s}^{R_c} \frac{G M_r \rho}{r^2}\, dr \simeq 2.3 \times 10^{11}\ atm$$

Equation of State, i.e., P=f(density, temperature, chemical composition)

Ideal gas law: $P = nKT$

n ← # density of particles, $n = N/V$

T ← temperature

For Hydrogen:

$$n = \frac{\langle \rho \rangle}{m_p} \qquad \langle \rho \rangle = \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3} \simeq 1 \ g/cm^3$$

NOTE: The equation of hydrostatic equilibrium with the equation of state of a perfect gas give us T_C and P_C, huge information! T_C is very large: the assumption of a perfect gas is correct, since H and He are completely ionized above 10^5K and many other elements are ionized above 10^6K. The very center of the Sun is a plasma (ions + electrons)

NOTE: If $\rho \ddot{r} \neq 0$, then there is no equilibrium between the force of gravity and pressure, hence **pulsations**

1) EQ. OF CONSERVATION OF MASS: $\frac{dM_r}{dr} = 4\pi r^2 \rho(r)$

2) EQ. OF HYDROSTATIC EQUILIBRIUM: $\frac{dP}{dr} = -G\frac{M_r \rho}{r^2}$

I need an equation of state to solve this set of equations, given by the

PRESSURE INTEGRAL: $P = \frac{1}{3}\int_0^\infty p\, v_p\, n(p)\, dp$

A) PERFECT, MONOATOMIC, NON-DEGENERATE GAS

For non-relativistic particles: p=mv
n(p)dp=n(v)dv

For relativistic particles: $v = \frac{p/m}{\sqrt{1+(p/mc)^2}}$

$$n(v)\,dv = n\left(\frac{m}{2\pi KT}\right)^{3/2} e^{-mv^2/2KT}\, 4\pi v^2 dv$$

Maxwell-Boltzmann distribution of velocities

$$P = \frac{1}{3}\int_0^\infty m v^2 n(v)\,dv = \frac{1}{3} m \underbrace{\int_0^\infty v^2 n(v)\,dv}_{:=\, n\overline{v^2}} \quad n\overline{v^2} = \frac{3KT}{m} n$$

$$\sqrt{\overline{v^2}} = v_{rms} = \sqrt{\frac{3KT}{m}}$$

$$\Rightarrow \quad P = \frac{1}{3} m \cdot n \frac{3KT}{m}$$

$$\Rightarrow \quad \boxed{P = nKT}$$

number density of particles

$$n = \frac{\rho}{\bar{m}} \leftarrow \text{average mass of gas particles}$$

Mean molecular weight of a perfect gas:

$$\mu := \frac{\overline{m}}{m_H} \qquad \text{w/ } m_H = 1.673532499 \times 10^{-24}\,g$$

$$\boxed{P = \frac{\rho k T}{\mu m_H}}$$

The mean molecular weight depends on the composition of the gas and on the ionization state of each species (free electrons must be included in the calculation of the average mass, hence I need to apply Saha's equation).

For neutral gas:

$$\overline{m}_n = \frac{\sum N_j m_j}{\sum N_j} \longrightarrow \mu_n = \frac{\overline{m}_n}{m_H} = \frac{\sum N_j A_j}{\sum N_j}$$

w/ $A_j := m_j / m_H$ mass number

For completely ionized gas:

For neutral gas:

$$\frac{1}{\mu_n} \simeq X + \frac{Y}{4} + \left\langle \frac{1}{A} \right\rangle_n Z \qquad \left\langle \frac{1}{A} \right\rangle_n \sim \frac{1}{15.5}$$

For completely ionized gas:

$$\frac{1}{\mu_i} \simeq 2X + \frac{3}{4}Y + \underbrace{\left\langle \frac{1+Z}{A} \right\rangle_i}_{\simeq 1/2} Z$$

$$X = 0.7 \quad Y = 0.28 \quad Z = 0.02 \Rightarrow \mu_n = 1.30 \quad \mu_i = 0.62$$

For electrons:

$$\frac{1}{\mu_e} := X(n_H - 1) + \frac{Y}{4}(n_{He} - 1) + (1 - X - Y)\underbrace{\left\langle \frac{n_Z - 1}{A_Z} \right\rangle}_{= 1/2}$$

$$n_H = 2$$
$$n_{He} = 3$$
$$n_Z = n_Z + 1$$

$$\rightarrow \begin{array}{l} n_H - 1 = 1 \\ n_{He} - 1 = 2 \\ n_Z - 1 = n_Z \end{array}$$

$$\mu_e|_i = \frac{2}{X+1}$$

For completely ionized gas

NOTE: from Maxwell-Boltzmann's equation:

$$\overline{V^2} = \frac{3KT}{m} \longrightarrow \frac{1}{2} m \overline{v^2} = \frac{3}{2} KT$$

$\Rightarrow$ average kinetic energy per particle per degree of freedom $= \frac{1}{2} KT$

NOTE: There are two extremely important physical cases for which the equation of state of a perfect, non-degenerate monoatomic gas is not adequate:

A) the pressure due to photons in the star interior is comparable to the pressure due to particles (radiation pressure P_{rad})
B) the electron gas becomes degenerate

$$P_{TOTAL} = P_{gas} + P_{rad} = \frac{\rho k T}{\mu m_H} + \frac{1}{3} a T^4$$

$$a = 7.565 \times 10^{-15} \frac{erg}{cm^3 \, {}^\circ K^4}$$

ELECTRON DEGENERACY:
Electrons obey the Fermi-Dirac statistics (they are 1/2 integer spin particles) —> Pauli exclusion principle

Maximum density of electrons in momentum space produces the degeneracy pressure. As the density increases, the electrons are force into higher-laying momentum states because the lower states are occupied, making a larger contribution to the pressure integral.

NON-RELATIVISTIC COMPLETE DEGENERACY:

$$P_{e,nr} = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3} = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{\rho}{\mu_e}\right)^{5/3}\left(\frac{1}{m_H}\right)^{5/3}$$

$$= 1.004 \times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3} \frac{\text{dynes}}{\text{cm}^2}$$

$$P_{e,nr} \propto n_e^{5/3} \propto \rho^{5/3}$$

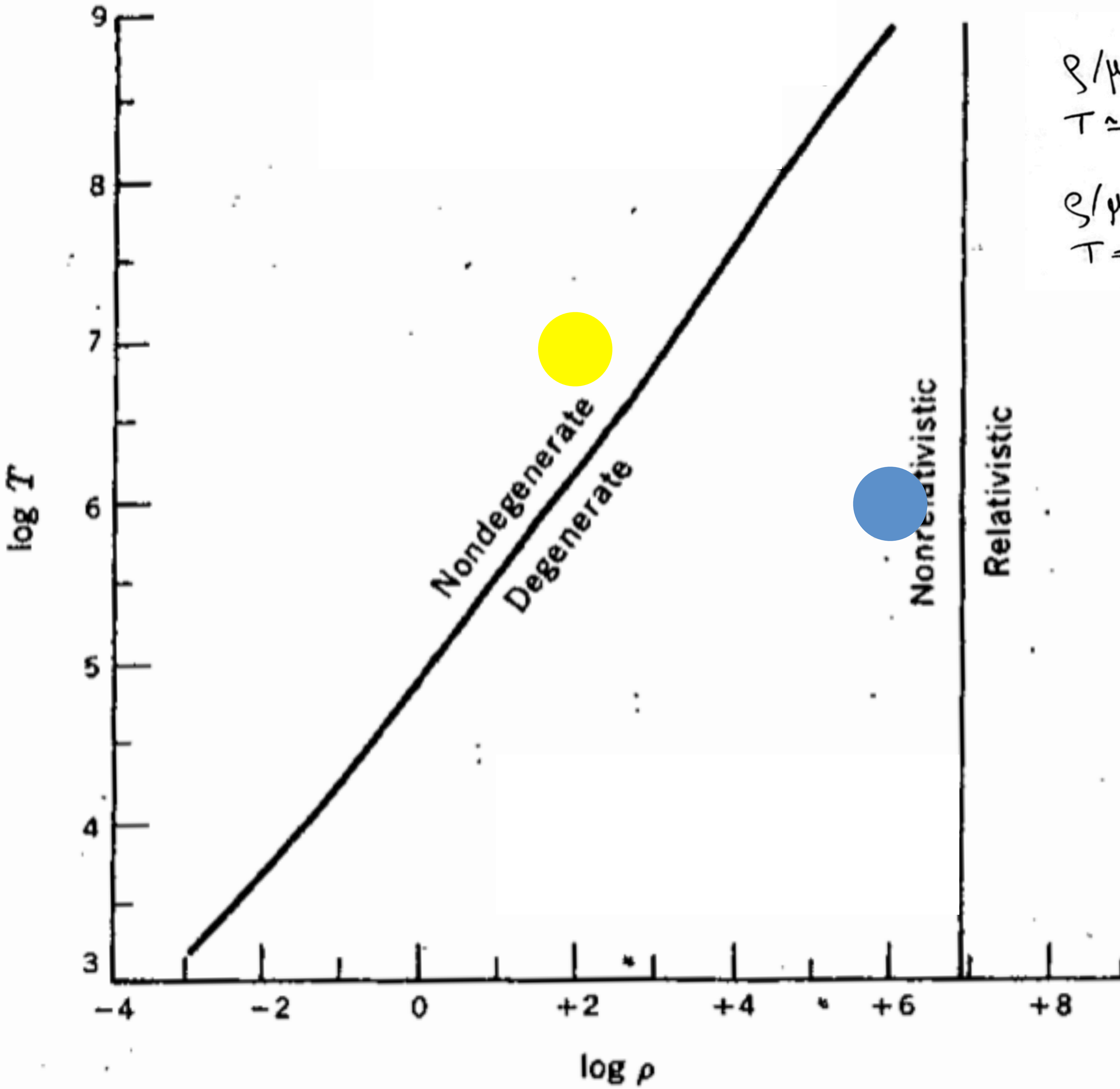
$$P_{\substack{\text{non-deg} \\ e^- \text{ gas}}} \propto n_e$$

$\Rightarrow$ AS n_e INCREASES $P_{e,nr} > P_{\substack{\text{non-deg} \\ e^- \text{ gas}}}$

SETTING $P_{\text{non deg}} = P_{e,nr}$ $\Rightarrow$ BOUNDARY IN THE $\log \rho$ vs $\log T$ PLOT

$$\frac{kT\rho}{\mu_e m_H} = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{\rho}{\mu_e}\right)^{5/3}\left(\frac{1}{m_H}\right)^{5/3}$$

$\Rightarrow$ $P_{e,nr} > P_{e,\text{non deg}}$ WHEN $\dfrac{\rho}{\mu_e} > 2.4 \times 10^{-8}\, T^{3/2}\ \dfrac{g}{cm^3}$



$\rho/\mu_e \simeq 10^2$ $\Rightarrow$ NON DEGENERATE
$T \simeq 10^7$ e^- PRESSURE

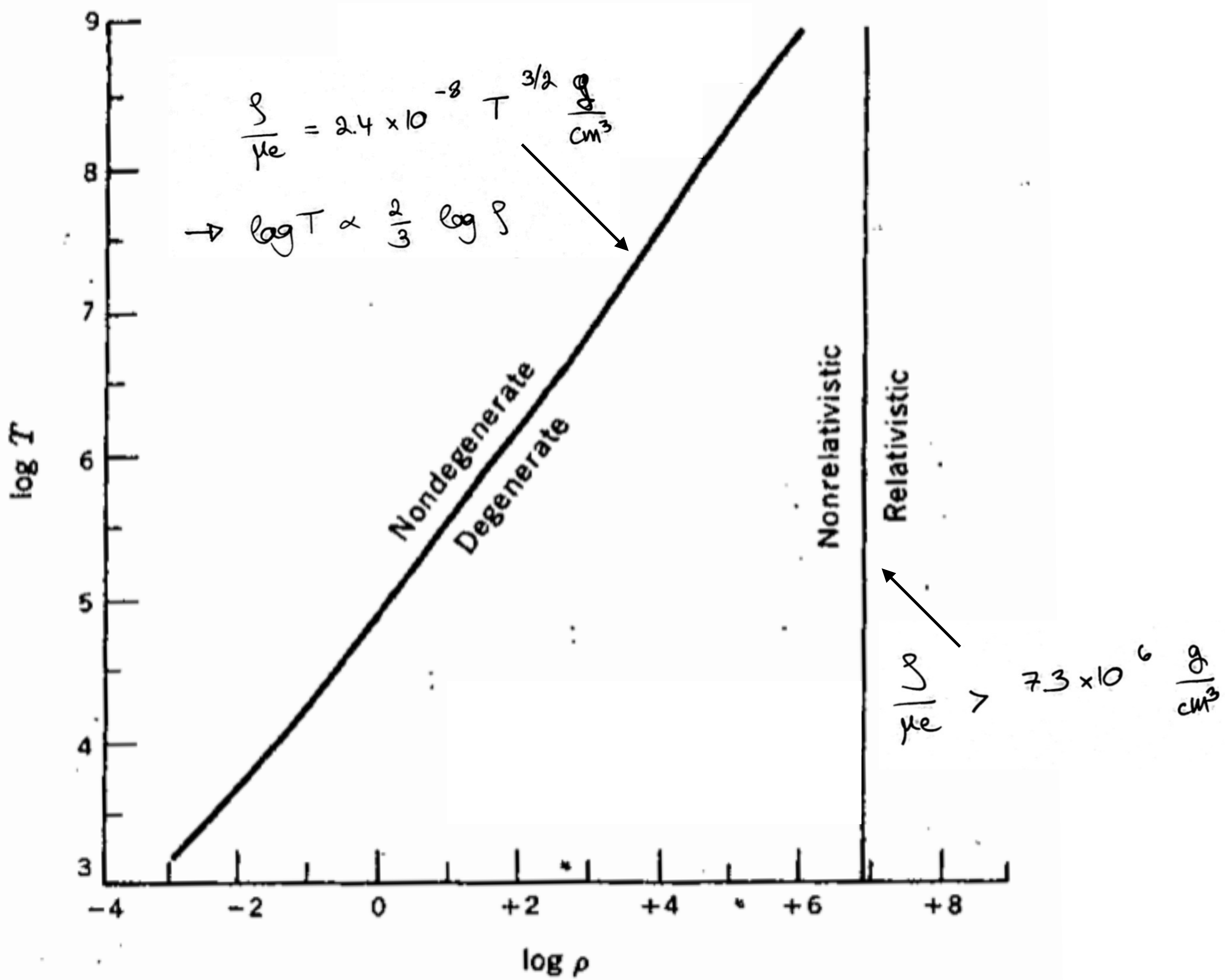
$\rho/\mu_e \simeq 10^6$ $\Rightarrow$ COMPLETE
$T = 10^6$ DEGENERATE
PRESSURE

RELATIVISTIC COMPLETE DEGENERACY:

As the electron density n_e increases, the minimum allowed momentum of electrons grows larger to the point that the electrons in the degenerate distribution become relativistic. This happens when: $\frac{\rho}{\mu_e} > 7.6 \times 10^6 \ \frac{g}{cm^3}$

$$P_{e,r} = \frac{ch}{8}\left(\frac{3}{\pi}\right)^{1/3} n_e^{4/3}$$

$$\text{ie:} \quad P_{e,r} \propto n_e^{4/3}$$



$\frac{\rho}{\mu_e} = 2.4 \times 10^{-8}\, T^{3/2}\, \frac{g}{cm^3}$
$\rightarrow \log T \propto \frac{2}{3} \log \rho$
Nondegenerate
Degenerate
Nonrelativistic
Relativistic
$\frac{\rho}{\mu_e} > 7.3 \times 10^{6}\, \frac{g}{cm^3}$
log T
log ρ
9
8
7
6
5
4
3
−4
−2
0
+2
+4
+6
+8

Regarding the mechanical pressure supporting the star $$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$$

$$P = P_e + P_{nuclei}$$

Nuclei are never degenerate in common stars (except neutron stars)

P_{nuclei} is that of a maxwellian gas, using the appropriate value of the mean molecular weight. Since electrons have been accounted for separately, use only the mean molecular weight of the remaining ions and nuclei

$$P_{gas} = P_e + \frac{\rho K T}{\mu_i m_H}$$

μ_i mean molecular weight of ions

In practical cases when electron degeneracy occurs, nuclei are generally He, C, O, and heavier nuclei. Therefore, P_e provides the bulk of the pressure, with P_{nuclei} only a small addition.

$$P_e \simeq n_e K T \, [1 + D]$$

correcting factor

$$D \propto \frac{n_e}{m^{3/2} T^{3/2}}$$

$$\frac{D_{nuclei}(n,p)}{D_{e^-}} = \left(\frac{m_e}{m_p}\right)^{3/2} \simeq 10^{-5}$$

i.e., when the density increases, D becomes non-negligible for electrons well before protons and neutrons.

NOTE: The pressure of a completely degenerate gas does not explicitly depends on T. Therefore, a small rise in T of an almost complete degenerate electron gas causes almost no change in the pressure. This is very crucial on stellar structure and stellar evolution (runaway in nuclear reaction rates, a.k.a., flash phenomena).

NOTE: When electrons are degenerate, heat conductivity becomes important. When non-degenerate, the mean free path of a charged particle is very small, hence conductivity is extremely inefficient at transporting energy. But when degenerate, the mean free path of electrons is quite long, because the exchange of energy among electrons is suppressed, making energetic electrons free to move about —> isothermal structure (same T).

$$P = P_{ions} + P_{e^-s} + P_{radiation}$$

If electron gas non degenerate: $P_{ions} + P_{e^-s} = P_{gas} = \dfrac{\rho K T}{\mu m_H}$ μ mean molecular weight for all particles

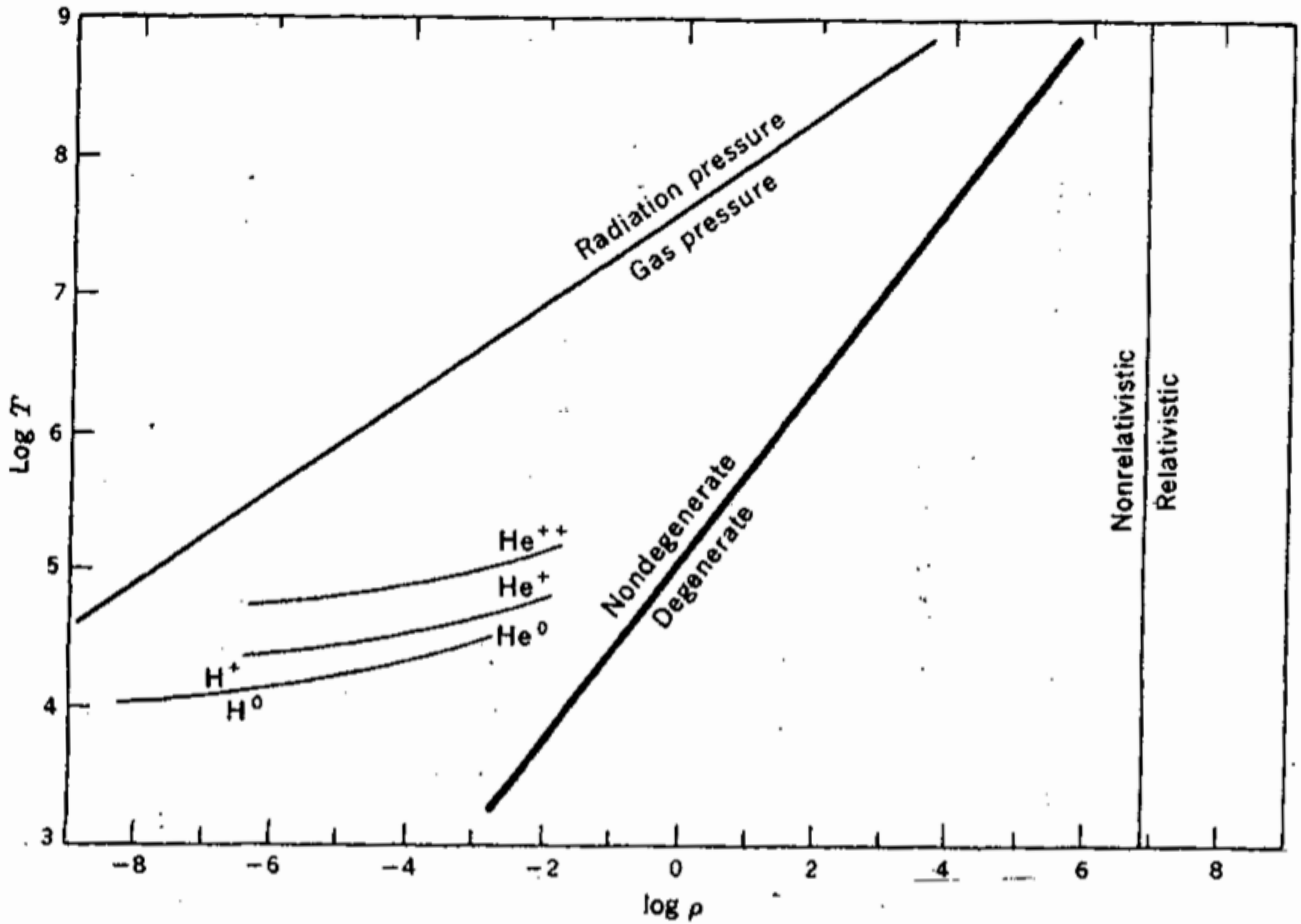
If electron gas degenerate: $P_{ions} = \dfrac{\rho K T}{\mu_{ions} m_H}$

$$P_e \propto \begin{cases} (\rho/\mu_e)^{5/3} & \text{Completely degenerate, non-relativistic} \\ (\rho/\mu_e)^{4/3} & \text{Completely degenerate, relativistic} \end{cases}$$

$$P_{gas} = P_{rad} \longrightarrow \frac{1}{3} a T^4 = \frac{\rho K T}{\mu m_H}$$

$$\Longleftrightarrow \quad T = 3.2 \times 10^7 \left(\frac{\rho}{\mu}\right)^{1/3} \simeq 3.6 \times 10^7 \rho^{1/3}$$

$$\Rightarrow \quad \log T \simeq \frac{1}{3} \log \rho + 7.56$$



Log T
9
8
7
6
5
4
3
log ρ
−8
−6
−4
−2
0
2
4
6
8
Radiation pressure
Gas pressure
Nondegenerate
Degenerate
Nonrelativistic
Relativistic
He⁺⁺
He⁺
He⁰
H⁺
H⁰

Stellar Structure

Reading assignment
THURSDAY 10/20: Chapter 10.3

Homework Assignment #3 due by:
TUESDAY 10/27 before beginning of class