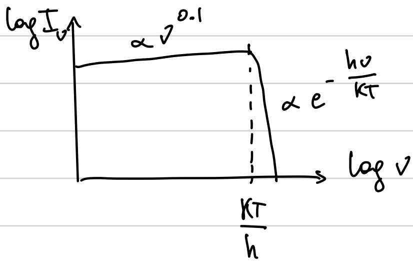
BREMSSTRAHLUNG EXAMPLES

A. GALAXY CLUSTERS

- $T \sim 10^7 - 10^8$ K $\rightarrow$ radiation emitted up to $\nu \sim 10^{19}$ Hz (X-ray)
- $n \sim 10^{-3} / cm^3$
- $R \sim 1$ Mpc
- SPECTRUM

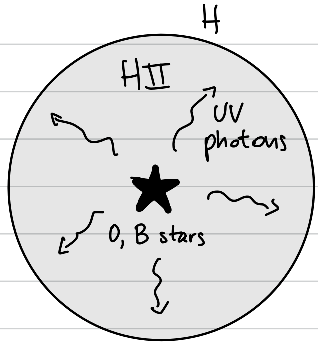


B. HII REGIONS

IONIZED HYDROGEN

- UV photons from O, B type stars (young, hot stars) ionize atoms
- $T \sim 1\text{-}2 \times 10^4$ K
- n : few $-$ $\gtrsim 10^6\ cm^{-3}$
- R : 100pc $-$ $\lesssim 1$ pc

  GIANT   COMPACT



- SPECTRUM : RADIO CONTINUUM

$$\begin{cases} j_\nu^{ff} \simeq 6.5 \times 10^{-38}\, Z^2 n_i n_e T^{-1/2} e^{-\frac{h\nu}{kT}} \bar{g}_{ff} \\ \alpha_\nu^{ff} \simeq 0.018\, Z^2 n_i n_e T^{-3/2} \nu^{-2} \bar{g}_{ff} \end{cases}$$

In radio :
$$\begin{cases} \bar{g}_{ff} \simeq 12\, \nu^{-0.1}\, T^{0.15} \\ e^{-\frac{h\nu}{kT}} \sim 1 \quad (h\nu \ll kT) \end{cases} \rightarrow \text{replacing in } j_\nu^{ff} \text{ and } \alpha_\nu^{ff} \rightarrow$$

$n_e = n_i$ in HII regions

$$\begin{cases} j_\nu^{ff} \sim 6.5\times10^{-38}\, n^2\, T^{-0.35}\, \nu^{-0.1} \\ \alpha_\nu^{ff} \sim 0.2\, n^2\, T^{-1.35}\, \nu^{-2.1} \end{cases}$$

RADIATIVE TRANSFER EQUATION: (common case with no background source)

$$I_\nu = \frac{j_\nu}{\alpha_\nu}\left(1-e^{-\tau_\nu}\right)$$

$$\simeq 3.2\times10^{-15}\, T_4\, \nu_{GHz}^2 \left[1-\exp(-\tau)\right]$$

where $n_2 = n/100\,cm^3$, $T_4 = T/10^4K$, $\nu_{GHz} = \nu/10^9 Hz$, $R_{10} = R/10pc$

$$\tau = \alpha_\nu R = 0.03\, n_2^2\, T_4^{-1.35}\, \nu_{GHz}^{-2.1}\, R_{10}$$

FLUX DENSITY:

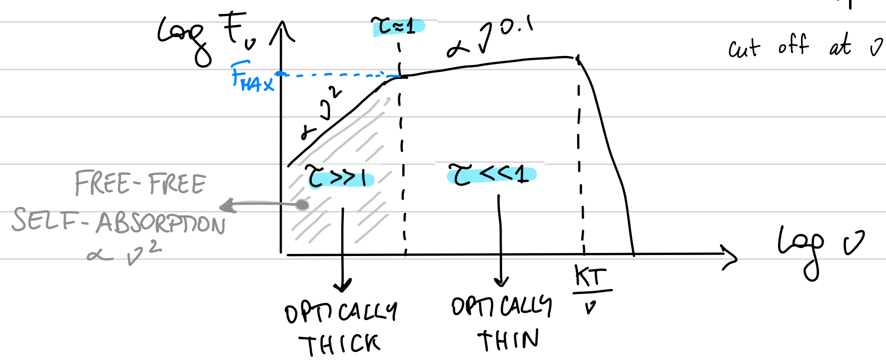
$$F_\nu = I_\nu \boxed{\frac{\pi R^2}{d^2}} \simeq 3.2\times10^{-15}\, T_4\, R_{10}^2\, d_{kpc}^{-2}\, \nu_{GHz}^2 \left[1-\exp\left(-0.03\, n_2^2\, T_4^{-1.35}\, \nu_{GHz}^{-2}\, R_{10}\right)\right]$$

solid angle for small sources

d: distance

$$\simeq 10^4\, R_{10}^2\, d_{kpc}^2\, \nu_{GHz}^2\, T_4 \left[1-\exp(-\tau)\right] \quad [Jy]$$

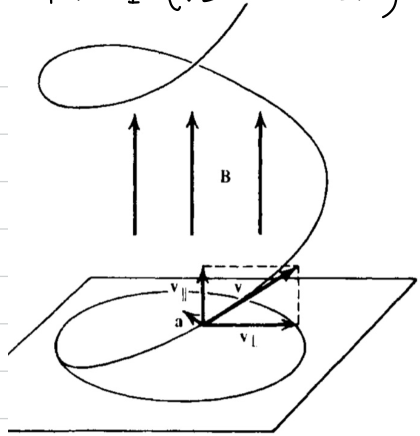
F_{max} when $\tau \simeq 1$: $F_{\nu,max} \sim 6.3\times10^4\, R_{10}^2\, d_{kpc}^{-2}\, \nu_{GHz}^2\, T_4$ [Jy]



cut off at $\nu \sim \frac{KT}{h} \sim 2\times10^{14}\, T_4$ [Hz]

SYNCHROTRON RADIATION

FIG 1 (R&L FIG 6.1)



Particles accelerated by a magnetic field $\bar{B}$ will radiate

NONRELATIVISTIC v	RELATIVISTIC v
↓	↓
CYCLOTRON RADIATION	SYNCHROTRON RADIATION
The frequency of emission is simply the frequency of gyration in the magnetic field.	The frequency spectrum is much more complex...

1. TOTAL POWER

Deriving the total power over frequencies and emission angles, emitted by a single e-, requires the generalization of the LARMOR FORMULA to the relativistic case.

LARMOR FORMULA $\quad P = \frac{2q^2a^2}{3c^3}$

Considering only electrons, the relativistic Larmor formula becomes:

$$P' = \frac{2e^2}{3c^3}\left[a_\perp'^2 + a_\parallel'^2\right]$$

acceleration components ∥ and ⊥ to the velocity.
NOTE: we consider the frame that is istantaneously at rest with the particle

The Lorentz transform of these two components of the acceleration are:

$$\begin{cases} a'_\parallel = \gamma^3 a_\parallel \\ a'_\perp = \gamma^2 a_\perp \end{cases}$$

where $\gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2} = \left(1-\beta^2\right)^{-1/2}$ is the Lorentz factor.

$$\Rightarrow P' = \frac{2e^2}{3c^3}\gamma^4\left[\gamma^2 a_{\parallel}^2 + a_{\perp}^2\right] \quad ⊛$$

NOTE: change of direction of velocity, not the modulus, brings to LARGE accelerations

Let's now calculate $a_{\parallel}$ and $a_{\perp}$. Let's consider the motion of the e^- in the $\bar{B}$ field:

$$\frac{d}{dt}(\gamma m \bar{v}) = \bar{F}_{LORENTZ} = e\left(\underbrace{\bar{E}}_{=0\text{ usually on large scale}} + \frac{\bar{v}}{c}\times\bar{B}\right)$$

$$F_L = \frac{d}{dt}(\gamma m\bar{v}) = e\frac{\bar{v}}{c}\times\bar{B}$$

$$F_{L\parallel} = e v_{\parallel} B = 0 \rightarrow \boxed{a_{\parallel} = 0} \Rightarrow v_{\parallel} = \text{constant}$$

$$F_{L\perp} = \gamma m \frac{dv_{\perp}}{dt} = \frac{e}{c} v_{\perp} B$$

$\Rightarrow$ The solution to this equation is UNIFORM CIRCULAR MOTION of the projected motion on the normal plane, since $\bar{a}$ in this plane is normal to $\bar{v}$ $\Rightarrow$ CIRCULAR MOTION + UNIFORM MOTION ALONG THE FIELD $\rightarrow$ HELICAL MOTION **(FIG 1)**

$$\Rightarrow \boxed{a_{\perp} = \frac{e\,v\,B\sin\alpha}{\gamma m c}} = v_{\perp}\,\omega_B$$

α: PITCH ANGLE between $\bar{v}$ & $\bar{B}$

$\beta = \frac{v}{c}$

ω_B — FREQUENCY OF ROTATION, OR GYRATION $\omega_B = \frac{eB}{\gamma m c}$

Larmor Radius of the circular part of the motion $r_L = \frac{v_{\perp}^2}{a_{\perp}} = \frac{\gamma m c^2 \beta \sin\alpha}{eB}$

Replacing $a_{\parallel} = 0$ and $a_{\perp}$ in the equation of total emitted radiation ⊛

$$\boxed{P = \frac{2e^4}{3c^3 m^2}\gamma^2 B^2 \beta^2 \sin^2\alpha}$$

Plugging in numbers for e^-, with $\beta \rightarrow 1 \Rightarrow P = 1.58\times10^{-15} B^2\gamma^2\beta^2\sin^2\alpha$

Using
- $U_B \equiv \dfrac{B^2}{8\pi}$ MAGNETIC ENERGY DENSITY
- $r_0 = \dfrac{e^2}{m_e c^2}$ CLASSICAL ELECTRON RADIUS
- $\sigma_T = \dfrac{8\pi r_0^2}{3} = 6.65 \times 10^{-25}\ cm^2$ THOMSON SCATTERING CROSS SECTION

$\Rightarrow P(\alpha) = 2\sigma_T c\, U_B \gamma^2 \beta^2 \sin^2\alpha$ TOTAL SYNCHROTRON POWER EMITTED BY A SINGLE e^- OF GIVEN PITCH ANGLE α

For isotropic distribution of velocities, we need to average the term $\underline{\sin^2\alpha}$ over solid angle. For a given speed β: $\langle \beta_\perp^2 \rangle = \dfrac{\beta^2}{4\pi}\int \sin^2\alpha\, d\Omega = \dfrac{2}{3}\beta^2$

$$\Rightarrow \boxed{\langle P \rangle = \frac{2}{3}\sigma_T c\, U_B \gamma^2 \beta^2} \quad [erg/s]$$

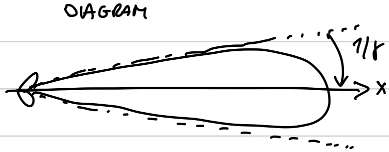
NOTE: $P \propto m^{-2} \rightarrow e^-$ radiate 1836^2 times more power than do protons of same γ.

2. SPECTRUM

1. relativistic e^-
2. $\vec{a}$ is $\perp$ to $\vec{v}$

$\Rightarrow$ THE EMISSION IS BEAMED

DIAGRAM



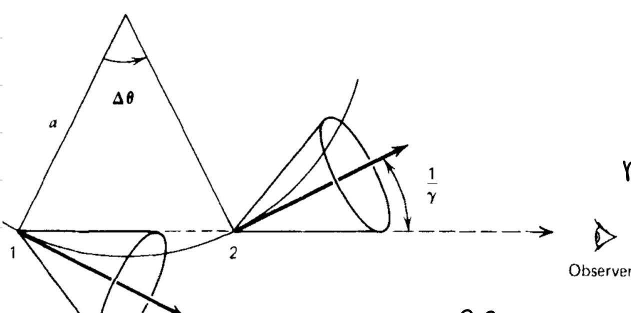


FIG 6.2 ON R&L

The observer will see a PULSE of radiation confined to a time interval much smaller than the gyration period, while the particle travels from point 1 to 2 along the line of sight.

RADIUS

From the radius of curvature of the path: $\boxed{a = \frac{\Delta s}{\Delta\theta}}$ PATH LENGHT along arc between 1 & 2

1) From geometry $\Delta\theta = \frac{2}{\gamma} \Rightarrow \Delta s = \frac{2a}{\gamma}$

2) From eq. of motion $\gamma m \frac{\Delta \bar{v}}{\Delta t} = \frac{q}{c} \bar{v} \times \bar{B}$

3) $|\Delta \bar{v}| = v \Delta\theta$ because v is constant

$\Delta s = v \Delta t$

replacing in the eq. of motion: $\frac{\Delta\theta}{\Delta s} = \frac{qB\sin\alpha}{\gamma m c v}$

$\Rightarrow a = \frac{\Delta s}{\Delta\theta} = \overbrace{\frac{v}{\omega_B}}^{= r_L} \frac{1}{\sin\alpha}$ which differs a factor $\sin\alpha$ from the Larmor radius r_L of the circle of projected motion in a plane normal to $\bar{B}$.

$$\Delta s = a \underbrace{\Delta\theta}_{2/\gamma} = \frac{2v}{\gamma \omega_B \sin\alpha}$$

EMITTING TIME during which the e^- emits radiation that will reach the observer:

$$\Delta t_e = t_2 - t_1 = \frac{\Delta s}{v} = \frac{2}{\gamma \omega_B \sin\alpha}$$

IN THE PARTICLE'S FRAME

times at which the particle passes points 1 & 2

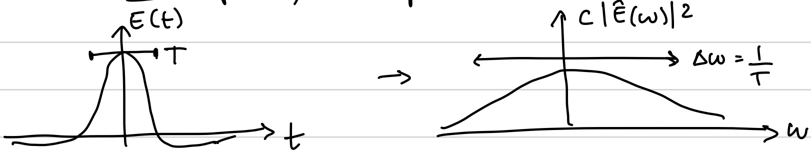
In the OBSERVER'S FRAME, the PULSE is detected over an ARRIVAL TIME interval Δt^A, that is SHORTER than Δt_e.

$$\Delta t^A = t_2^A - t_1^A = \overbrace{\frac{\Delta s}{v}}^{\Delta t_e} (1-\beta) = \frac{2}{\gamma \omega_B \sin\alpha}(1-\beta)$$

since $1-\beta = \frac{1-\beta^2}{1+\beta} = \frac{1}{\gamma^2(1+\beta)} \underset{\beta\to 1}{\approx} \frac{1}{2\gamma^2} \Rightarrow$

$$\Rightarrow \Delta t^A \approx \frac{1}{\gamma^3 \omega_B \sin\alpha}$$ the width of the observed pulse is smaller than the gyration period by a factor γ^3

(From Section 2.3) From when we studied the spectrum associated with pulses, if we have <u>small</u> pulses, the spectrum will be broader



Let's define the <u>critical frequency</u> $\boxed{\omega_c \equiv \frac{3}{2}\gamma^2 \omega_B \sin\alpha}$

The spectrum will extend to something of order ω_c before falling off.

To derive the FULL SPECTRUM of synchrotron radiation, let's perform a Fourier transform: $\hat{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)\, e^{i\omega t}\, dt$, remembering that $\hat{E}(\omega)$ contains all the info about frequency behavior of $E(t)$. To convert it into frequency info we write the energy per unit time per unit area:

$$\frac{dW}{dt\,dA} = \frac{c}{4\pi} E^2(t) \Rightarrow \frac{dW}{dA\,d\omega} = c\,|\hat{E}(\omega)|^2$$ TOTAL ENERGY PER AREA PER FREQUENCY RANGE IN THE ENTIRE PULSE

Let's demonstrate that the power will be of the form $P(\omega) = C_1 F\left(\frac{\omega}{\omega_c}\right)$

$$\frac{dP}{dA\,d\omega} = \frac{dW}{dt\,dA\,d\omega},$$ if the pulse repeats on an average timescale we have:

$$= \frac{1}{T}\frac{dW}{dA\,d\omega} = \frac{c}{T}|\hat{E}(\omega)|^2$$

The pulse in the considered case repeats with a $\boxed{\text{period } \frac{2\pi \sin^2\alpha}{\omega_B}}$ in the OBSERVER'S FRAME.

$$\Rightarrow \frac{dP}{dA\,dW} = \frac{c\,\omega_B}{2\pi \sin^2\alpha} \left|\hat{E}(\omega)\right|^2$$

Writing now the power in unit of SOLID ANGLE and replacing $|\hat{E}(\omega)|^2$ with RADIATED ENERGY as we saw in chapter 3 for moving charges:

$$\hookrightarrow E(t) = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \left[\hat{n} \times \{ (\hat{n}-\bar{\beta}) \times \dot{\bar{\beta}} \} \kappa^{-3} \right] e^{i\omega t}\, dt \right|^2$$

$$\kappa \equiv 1 - \hat{n}\cdot\bar{\beta}, \quad \hat{n} \text{ unit vector} = \frac{\bar{R}}{R}, \quad \bar{R}(t') = \bar{r} - r_0(t)$$

then, we have:

$$\frac{dP}{d\omega\, d\Omega} = \frac{e^2 \omega_B}{8\pi^3 c} \left| \int_{-\infty}^{\infty} \left[\hat{n} \times \{ (\hat{n}-\bar{\beta}) \times \dot{\bar{\beta}} \} \kappa^{-3} \right] e^{i\omega t}\, dt \right|^2$$

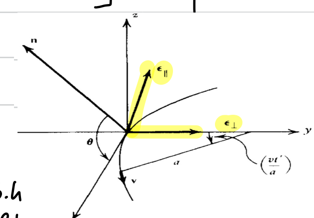
We measure κ, $\bar{\beta}$, $\dot{\bar{\beta}}$ in retarded frame, so let's change integration variable to $t' = t - \frac{R(t')}{c}$, $dt = \kappa\, dt'$, and approximate $\bar{R}' = |\bar{r}| - \hat{n}\cdot\bar{r}_0$, as we already did before for the general case in ch3.

Then:

$$\frac{dP}{d\omega\, d\Omega} = \frac{e^2 \omega_B}{8\pi^3 c} \left| \int_{-\infty}^{\infty} \underbrace{\hat{n} \times \left[(\hat{n}-\bar{\beta}) \times \dot{\bar{\beta}} \right] \kappa^{-2}}_{= \frac{d}{dt}\left[\frac{\hat{n} \times (\hat{n} \times \bar{\beta})}{\kappa} \right]} \exp\left[\frac{i\omega}{c} \left(t' - \hat{n}\cdot\bar{r}_0(t') \right) \right] dt' \right|^2$$

$$\Rightarrow (A)\quad \frac{dP}{d\omega\, d\Omega} = \frac{e^2 \omega^2 \omega_B}{8\pi^3 c} \left| \int_{-\infty}^{\infty} \hat{n} \times (\hat{n} \times \beta) \exp\left[\frac{i\omega}{c} \left(t' - \hat{n}\cdot\bar{r}_0(t') \right) \right] dt' \right|^2$$

We want to write (A) in the two polarization states $\varepsilon_\perp$, $\varepsilon_\parallel$



see Fig 6.4 on R&L

$$\frac{dP}{d\omega d\Omega} = \frac{e^2\omega^2\omega_B}{8\pi^3 c}\left|\int_{-\infty}^{\infty} \underbrace{\hat{n}\times(\hat{n}\times\bar{\beta})}_{\star} \exp\left[\frac{i\omega}{c}\underbrace{\left(t' - \hat{n}\cdot\bar{r}_0(t')\right)}_{\star\star}\right] dt'\right|^2$$

$\star$ $$\hat{n}\times(\hat{n}\times\bar{\beta}) = -\bar{\varepsilon}_\perp \sin\left(\frac{vt'}{a}\right) + \bar{\varepsilon}_\parallel \cos\left(\frac{vt'}{a}\right)\sin\theta$$

For short time interval such that $\frac{vt'}{a} \ll 1$, $\theta \ll 1$ because of beaming,

and $|\bar{\beta}| = 1$. Expanding sin and cos, and ignoring small cubic terms, we get:

$$\hat{n}\times(\hat{n}\times\bar{\beta}) \approx -\bar{\varepsilon}_\perp\left(\frac{ct'}{a}\right) + \bar{\varepsilon}_\parallel\theta \qquad \left[\begin{array}{l} \sin x = x - \frac{x^3}{6} + \dots \\ \cos x = 1 - \frac{x^2}{2} + \dots \end{array}\right]$$

$\star\star$ $$t' - \frac{\hat{n}\cdot\bar{r}(t')}{c} \approx t' - \frac{a}{c}\cos\theta\sin\left(\frac{v't}{a}\right)$$

Expanding sin and cos for small arguments, using again the approximation $1 - \frac{v}{c} \approx \frac{1}{2\gamma^2}$ and $v = c$, we have:

$$t' - \frac{\hat{n}\cdot\bar{r}_0(t')}{c} \approx (2\gamma^2)^{-1}\left[(1+\gamma^2\theta^2)t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right]$$

We can now finally write eq. (A) in the 2 polarization directions.
Expanding sin and cos again, and defining $\theta_\gamma \equiv 1 + \gamma^2\theta^2$

$$\frac{dP}{d\omega d\Omega} = \frac{dP_\parallel}{d\omega d\Omega} + \frac{dP_\perp}{d\omega d\Omega}$$

$$\frac{dP_\parallel}{d\omega d\Omega} = \omega_B \frac{q^2\omega^2\theta^2}{4\pi^2 c}\left|\int \exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right] dt'\right|^2$$

$$\frac{dP_\perp}{d\omega d\Omega} = \omega_B \frac{q^2\omega^2}{4\pi^2 c}\left|\int \frac{ct'}{a} \exp\left[\frac{i\omega}{2\gamma^2}\left(\theta_\gamma^2 t' + \frac{c^2\gamma^2 t'^3}{3a^2}\right)\right] dt'\right|^2$$

Let's change variables: $y \equiv \frac{\gamma c t'}{a\theta_\gamma}$, $\eta \equiv \frac{\omega a \theta_\gamma^3}{3c\gamma^3}$, then:

$$\frac{dP_{\parallel}}{d\omega d\Omega} = \frac{e^2\omega^2\omega_B\theta}{8\pi^3 c}\left(\frac{a\theta_\gamma}{\gamma_c}\right)^2 \left|\int_{-\infty}^{\infty} \exp\left[\frac{3}{2}i\eta\left(y+\frac{y^3}{3}\right)\right]dy\right|^2$$

$$\frac{dP_{\perp}}{d\omega d\Omega} = \frac{e^2\omega^2\omega_B}{8\pi^3 c}\left(\frac{a\theta_\gamma^2}{\gamma_c^2}\right)^2 \left|\int_{-\infty}^{\infty} y\exp\left[\frac{3}{2}i\eta\left(y+\frac{y^3}{3}\right)\right]dy\right|^2$$

Only LITTLE ERROR is made in extending the limits of integration from $-\infty$ to $+\infty$, instead of over time of pulse, since power is small before/after pulse.

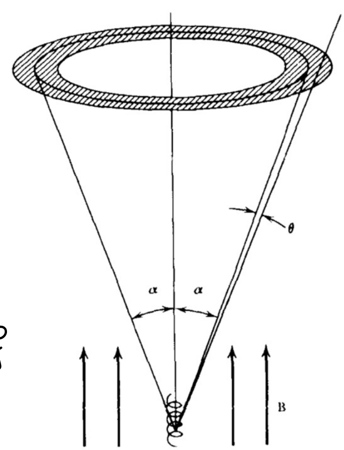
The integrals above are functions of η. Since the most radiation occurs at $\theta \simeq 0 \Rightarrow \eta = \eta|_{\theta=0} = \frac{\omega}{2\omega_c}$, so $P \propto F\left(\frac{\omega}{\omega_c}\right)$ as mentioned qualitatively above.

These integrals can be expressed in terms of the modified Bessel functions of 1/3 and 2/3 order:

$$\frac{dP_{\parallel}}{d\omega d\Omega} = \frac{e^2\omega^2\omega_B\theta^2}{3\pi^2 c}\left(\frac{a\theta_\gamma}{rc}\right)^2 K^2_{1/3}(\eta)$$

$$\frac{dP_{\perp}}{d\omega d\Omega} = \frac{e^2\omega^2\omega_B}{3\pi^2 c}\left(\frac{a\theta_\gamma^2}{rc}\right)^2 K_{2/3}(\eta)$$

Integrating over solid angle gives the power emitted by the particle per complete orbit in the projected normal plane. $d\Omega \simeq 2\pi \sin\alpha\, d\theta$, since in one gyration the particle traces out a cone of opening half-angle α and thickness $d\theta$.



R&L FIG 6.5

See K.C. Westfold 1959, APJ, 130, 241 to get:

$$\frac{dP_{\parallel}}{d\omega} = \frac{\sqrt{3}\, e^3 B}{4\pi m c^2} \left[F(x) - G(x) \right]$$

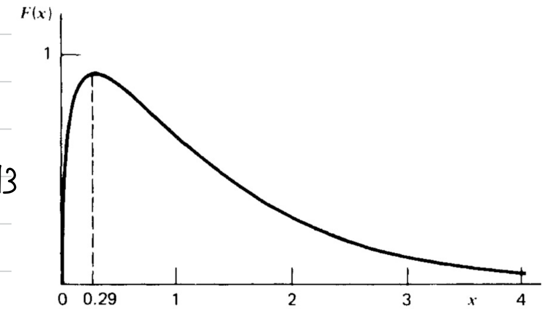
$$\frac{dP_{\perp}}{d\omega} = \frac{\sqrt{3}\, e^3 B}{4\pi m c^2} \left[F(x) + G(x) \right]$$

where $x \equiv \frac{\omega}{\omega_c}$, $\omega_c \equiv \frac{3}{2} \gamma^2 \omega_B \sin\alpha$

$$F(x) \equiv x \int_x^{\infty} K_{5/3}(z)\, dz \quad , \quad G(x) \equiv x\, K_{2/3}(x)$$

The TOTAL POWER EMITTED is the sum over the two polarization states:

$$\frac{dP}{d\omega} \simeq \frac{\sqrt{3}\, e^3 B \sin\alpha}{2\pi m c^2} F(x)$$



LOW FREQUENCIES:

$$x \ll 1 \rightarrow F(x) \sim \frac{4\pi}{\sqrt{3}\, \Gamma\left(\frac{1}{3}\right)} \left(\frac{x}{2} \right)^{1/3}$$

HIGH FREQUENCIES:

$$x \gg 1 \rightarrow F(x) \sim \left(\frac{\pi}{2} \right)^{1/2} x^{1/2} e^{-x}$$

The spectrum rises as frequencies to the 1/3 power at low x, and drops off exponentially at high x.

PEAK of spectrum at $x = 0.29\, \omega_c$.

3. POLARIZATION

The polarization is elliptical, which correspond to a combination of LINEAR + CIRCULAR polarization.

For a distribution of particles with random pitch angles α, the circular polarization nearly cancels such that $\Pi_c \sim \frac{1}{\gamma_{(\omega=\omega_c)}}$

LORENTZ FACTOR OF A PARTICLE WHOSE $\omega_c = \omega$ OF OBSERVATION

If $\bar{B}$ is uniform in direction throughout the source, the linear polarization for a single particle is:

$$\Pi_L(\omega) = \frac{P_\perp(\omega) - P_\parallel(\omega)}{P_\perp(\omega) + P_\parallel(\omega)} = \frac{G(x)}{F(x)}$$ P

with x along $\hat{\varepsilon}_\perp$, which is $\perp$ to $\bar{B}$'s direction as projected on sky

4. SYNCHROTRON SELF-ABSORPTION

All emission processes have their absorption counterpart, as we saw already with free-free thermal bremsstrahlung. In that case, we used the Kirchoff's law do derive the absorption coefficient. In the synchrotron case, we cannot do that, as we have 1) relativistic particles and 2) the particles distribution is not thermal. So, in this case we need to use relations between the A and B Einstein's coefficients relating spontaneous and stimulated emission and absorption.
The synchrotron absorption is between continuum states defined by the e^-'s momentum and position. To apply the Einstein's formalism, which was for transitions between discrete states, we need to discretize the continuum phase-space into elements of size h^3 (as per the Planck's principle) and treat transitions between these states as being discrete states.
To derive α_ν we sum over all possible upper (E_2) and lower (E_1) states:

$$\text{(A)} \quad \alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \phi_{21}(\nu)$$

EINSTEIN B-COEFFICIENTS: transition probability for absorption / stimulated emission

LINE PROFILE FUNCTION is a δ-function that restrict the value $E_2 = E_1 + h\nu$

We have assumed ISOTROPIC emission and absorption, which is true only if $\bar{B}$ is tangled and random in direction, and the particle distributions are isotropic.
We now want to reduce (A) to a form depending only on $P(\omega)$.
It's easier to write the emission in terms of ν rather than ω:

$$P(\nu, E_2) = 2\pi P(\omega)$$

E_2: energy of the radiating e^-

In terms of Einstein coeff:

$$P(\nu, E_2) = \frac{4\pi j_\nu}{n(E_2)} = h\nu \sum_{E_1} A_{21} \phi_{21}(\nu) = \left(\frac{2h\nu^3}{c^2}\right) h\nu \sum_{E_1} B_{21} \phi_{21}(\nu)$$

This includes the part of absorption due to stimulated emission.

For the absorption-only part we can write:

$$\frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left(n(E_1) B_{12}\right) \phi_{21}(\nu) = \frac{c^2}{8\pi h\nu^3} \sum_{E_2} \left[n(E_2 - h\nu) B_{12} \right] P(\nu, E_2)$$

possible because $\phi_{21}(\nu)$ acts like a δ-function so that $\sum_{E_1} (n(E_1)) \phi_{21}(\nu) \rightarrow n(E_2 - h\nu)$

Therefore:

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \sum_{E_2} \left[n(E_2 - h\nu) - n(E_2) \right] P(\nu, E_2)$$

For synchrotron radiation: $P(\nu, E_2) = 2\pi \frac{dP}{d\omega} \simeq \frac{\sqrt{3}\, e^3 B F(x)}{mc^2}$

Let's consider a population of electrons with an isotropic distribution of momentum p, such that $f(p)\, d^3p = 4\pi f(p) p^2 dp$

$f(p)$: e^- distribution function

$4\pi f(p) p^2 dp$: number of e^- per unit volume with momentum between $p + dp$

Number of quantum states per unit phase volume d^3p: $2/h^3$ (2 is the statistical weight for a spin-$\frac{1}{2}$ particle).

ELECTRON DENSITY PER QUANTUM STATE $n(E_2) \longrightarrow \frac{h^3}{2} f(p)$, $\sum_{E_2} \rightarrow \frac{8\pi}{h^3} \int p^2 dp$

Usually, we use Energy instead of momentum. Let's consider:

1. e^- distribution is isotropic
2. extreme relativistic case $E \simeq pc$
3. for an energy distribution of e^- $N(E)dE = 4\pi f(p) p^2 dp$

We then have: $$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int_{E_{min}}^{E_{max}} P(\nu,E)\left[\frac{N(E-h\nu)}{(E-h\nu)^2} - \frac{N(E)}{E^2}\right]E^2\,dE \quad (B)$$

Since absorption is low-frequency phenomenon and $dE \propto d\nu$, we can rewrite (B) in a more compact way:

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2}\int_{E_{min}}^{E_{max}} P(\nu,E)\,\frac{\partial}{\partial E}\left[\frac{N(E)}{E^2}\right]E^2\,dE$$

5. POWER-LAW DISTRIBUTION OF e⁻ ENERGIES

Cosmic sources accelerate e⁻ to high relativistic energies tend to produce POWER-LAW energy distributions: $\underbrace{N(E)}_{\text{number density of e}^- \text{ with energies between } E \text{ and } E+dE} = KE^{-s}$ over a wide range of energies.

If the power law extends from E_{min} to E_{max} and is $\emptyset$ otherwise, the density of relativistic e⁻ is given by:

$$n_{re} = \int_{E_{min}}^{E_{max}} KE^{-s}\,dE = \frac{K}{s-1}\left[E_{min}^{-(s-1)} - E_{max}^{-(s-1)}\right]$$

energy density $$u_{re} = \int_{E_{min}}^{E_{max}} KEE^{-s}\,dE = \frac{K}{s-2}\left[E_{min}^{-(s-2)} - E_{max}^{-(s-2)}\right]$$

The difference in the brackets is $= \ln\left(\frac{E_{max}}{E_{min}}\right)$ if $s=1$ in n_{re} and $s=2$ in u_{re}.

The EMISSION COEFFICIENT is : $j_\nu = \frac{1}{4\pi}\int_0^\infty N(E)\frac{dP}{d\nu}dE$ $(\nu_L \leq \nu \leq \nu_c)$

From the previous expression $\frac{dP_{\parallel\perp}}{d\nu} = 2\pi\frac{dP_{\parallel\perp}}{d\omega} \simeq \frac{\sqrt{3}\,e^3 B}{2mc^2}\left[F(x) \mp G(x)\right]$

with $x \equiv \frac{\nu}{\nu_c}$, so the critical density $\nu_c = \frac{\omega_c}{2\pi} = \frac{3e}{4\pi m^3 c^5} BE^2 \sin\Psi$
$= 6.27 \times 10^{18}\, BE^2 \sin\Psi$ [Hz]

where Ψ is the angle between $\bar{B}$ and the line of sight, which is also the pitch angle of e⁻ that beam radiation into the line of sight.

We have then: $j_{\nu,\parallel\perp} = \underline{c_1(s)}\, K(B\sin\Psi)^{(s+1)/2}\, \nu^{-(s+1)/2}$ C

combination of numerical factors (see TABLE)

From equation P and C, the linear polarization is:

$\Pi = \frac{3(s+1)}{3s+7}$ if $\bar{B}$'s direction is uniform throughout the source

For typical s ranges [~1.4 to ~3], $\Pi \sim$ [0.64 - 0.75]

The ABSORPTION COEFFICIENT is $\alpha_\nu = \underline{c_2(s)}\, K(B\sin\Psi)^{\frac{s+2}{2}}\, \nu^{-\frac{(s+4)}{2}}$ D

see TABLE

As for j_ν, α_ν is also different for the 2 polarizations.

If $\bar{B}$ in the source is extremely tangled (e.g. from turbulence), one can average $\sin\Psi$ by integrating over Ω over all directions and normalizing. The result from the integral is a ratio of 2 Γ functions of different numbers. See the table for values of $\langle \sin\Psi \rangle$.

Table of functions of slope s of the electron energy distribution

s	τ_m	c_1	c_2	$\langle(\sin\psi)^{(s+1)/2}\rangle$	$\langle(\sin\psi)^{(s+2)/2}\rangle$
1.5	0.25	1.01×10^{-18}	2.29×10^{12}	0.75	0.69
2.0	0.48	3.54×10^{-14}	1.17×10^{17}	0.72	0.67
2.5	0.69	1.44×10^{-9}	6.42×10^{21}	0.69	0.64
3.0	0.88	6.3×10^{-5}	3.5×10^{26}	0.67	0.62

For values of s not listed here, logarithmic interpolation will give reasonable approximations to c_1

So, e.g., $\log c_1(s=1.7) \approx \log c_1(s=1.5)] + \frac{1.7-1.5}{2.0-1.5}[\log c_1(s=2.0) - \log c_1(s=1.5)] = -16.17$

$\rightarrow c_1(s=1.7) \approx 10^{-16.178} = 6.6\times10^{-17}$

RADIATIVE TRANSFER OF SYNCHROTRON RADIATION FOR A POWER-LAW ENERGY DISTRIBUTION.

$$I_\nu = \frac{j_\nu}{\alpha_\nu}\left(1-e^{-\tau_\nu}\right)$$

I_ν has a peak at frequency ν_m corresponding to optical depth $\tau_m(s)$, which is a function of the slope. τ_m can be determined by setting $\frac{dI}{d\nu}=0$ and solving for τ_ν numerically (see Table).

- $\boxed{\nu << \nu_m}$, $\tau_\nu >> 1$ -> SOURCE IS OPTICALLY THICK

$$I_\nu(\nu<<\nu_m) = \frac{j_\nu}{\alpha_\nu} = \frac{c_1(s)}{c_2(s)}\left(B\sin\psi\right)^{-1/2}\nu^{5/2} \quad [\text{erg/s/cm}^3\text{/Hz/sr}]$$

FLUX DENSITY $F = I_\nu \times \Omega = I \times \frac{\pi R^2}{d^2}$ for uniform spherical source or face-on disk of radius R

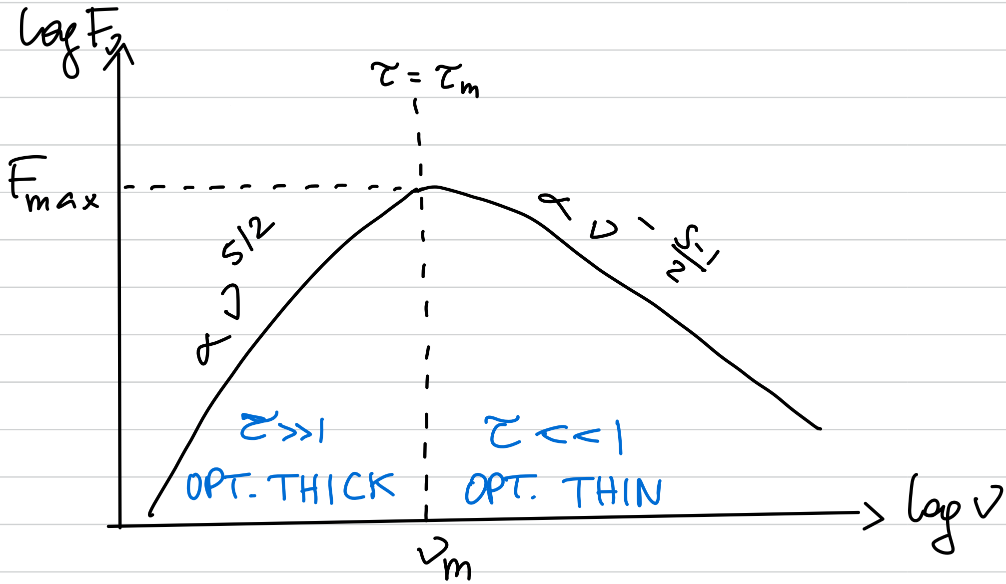
• $\boxed{\nu >> \nu_m}$, $\tau << 1 \longrightarrow$ SOURCE IS OPTICALLY THIN

$$I_\nu(\nu >> \nu_m) = X \, j_\nu = X \, c_1(s) K (B \sin\psi)^{\frac{s+1}{2}} \, \nu^{-\frac{s-1}{2}}$$

[erg/s/cm^3/Hz/sr]

X: path lenght through the source

Optical depth at ν_m: $\tau_m(s) = c_2(s) \, X \, K (B \sin\psi)^{\frac{s+2}{2}} \, \nu_m^{-\frac{s+4}{2}}$



$$F_\nu = I_\nu \times \Omega \quad \text{if source is uniform}$$

$$= \int_\Omega I_\nu \, d\Omega \quad \text{if source is non-uniform}$$

For an uniform spherical source of radius R and distance d, one should integrate over different path lenghts, BUT A REASONABLE APPROXIMATION IS TO ADOPT R AS THE TYPICAL PATH LENGHT.

$$F_{max} \approx \frac{4\pi}{3} C_1(s)\, d^{-2} R^3 K (B \sin\psi)^{\frac{s+1}{2}} \nu_m^{-\frac{s-1}{2}} e^{-\tau_m(s)}$$

$$F(\nu \gg \nu_m) \approx \frac{4\pi}{3} C_1(s)\, d^{-2} R^3 K (B \sin\psi)^{\frac{s+1}{2}} \nu_m^{-\frac{s-1}{2}}$$

LINEAR POLARIZATION FOR A POWER-LAW ENERGY DISTRIBUTION

- $\nu \gg \nu_m$: $\Pi = \frac{3(s+1)}{3s+7}$ as discussed before

  $\approx 70\%$ for an uniform $\bar{B}$. The position angle is $\perp$ to $\bar{B}$'s direction as projected on the sky.

- $\nu \ll \nu_m$: $\Pi = \frac{3}{6s+13} \approx 10-14\%$ for typical values of s

  The position angle is $//$ $\bar{B}$ direction.