FUNDAMENTALS OF RADIATIVE TRANSFER ≈ CH. 1

$\lambda\nu = c$ $\quad c = 3\times10^{10}$ cm/s free space speed of light

$E = h\nu$ $\quad h = 6.625\times10^{-27}$ erg s Planck's constant

$T = E/K$ $\quad$ temperature associated to each wavelength

$K = 1.38\times10^{-16}$ erg/°K Boltzmann's constant

$p = E/c$ $\quad$ momentum of a photon

Energy Flux F : consider an element of area dA exposed to radiation for a time dt. The amount of energy passing through the element should be proportional to dAdt

$\Rightarrow$ $F\,dA\,dt$ = energy passing through dA

$$[F] = \frac{\text{erg}}{\text{cm}^2\,\text{s}}$$

ISOTROPIC radiation: if it emits energy equally in all directions

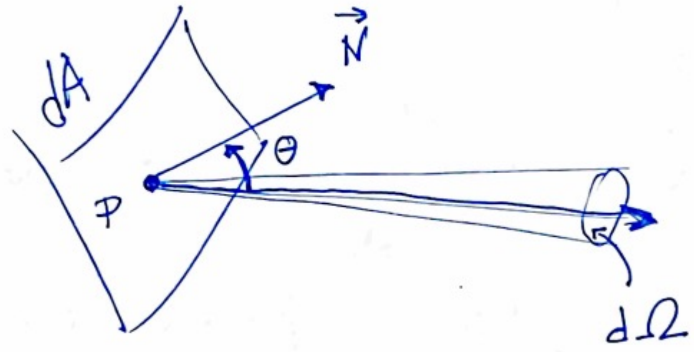
Conservation of energy $\longrightarrow$ $F(r_1)\,4\pi r_1^2 = F(r)\,4\pi r^2$

energy passing through imaginary spherical surface S_1 & S

$$\Rightarrow F(r) = \frac{F(r_1)\,r_1^2}{r^2} \Rightarrow \boxed{F = \frac{\text{constant}}{r^2}}$$

INVERSE SQUARE LAW

NOTE: The flux is a measure of the energy carried by all rays passing through a given area. A more detailed description of the radiation is by using the specific intensity (or brightness), ie: the energy carried along by a set of individual rays w/in the frequency range ν & $\nu+d\nu$.



$\vec{PO}$ direction of observer
$\vec{N}$ normal vector to dA
θ angle between $\vec{N}$ & $\vec{PO}$
$d\Omega$ solid angle
→ observer O

The energy emitted by dA in the unit of time, in the solid angle $d\Omega$ around the direction $\vec{PO}$ in the frequency interval $d\nu$ is:

$$dE_\nu = I_\nu(\theta) \underbrace{\cos\theta}_{} \, dA \, d\Omega \, d\nu \, dt$$

($dE = I_\nu \, dA \, dt \, d\Omega \, d\nu$)

↑ to have the component along $\vec{PO}$ (ie, we are interested only in the photons arriving perpendicularly to our detector)

$\boxed{I_\nu(\theta)}$ is the SPECIFIC INTENSITY OF THE RADIATION (or BRIGHTNESS) which depends on location in space, on direction, & on frequency.

$$[I_\nu(\theta)] = \frac{erg}{cm^2 \, s \, Hz \, str}$$

$$[F_\nu] = \frac{erg}{s \, cm^2 \, Hz}$$

ENERGY FLUX (or NET FLUX)

$$\boxed{F_\nu = \int I_\nu(\theta) \cos\theta \, d\Omega}$$

($dF_\nu = I_\nu \cos\theta \, d\Omega$)

Note: If I_ν is an isotropic radiation field (not a function of angle), then $F_\nu = 0$ since $\int \cos\theta \, d\Omega = 0$, ie: there is just as much energy crossing dA in the direction $\vec{N}$ as the $-\vec{N}$ direction.

In the assumption of spherical symmetry

$$d\Omega = 2\pi \sin\theta \, d\theta \qquad \Big(d\Omega = \sin\theta \, d\theta \, d\phi \quad \phi \in [0, 2\pi] \quad \theta \in [0, \pi/2] \text{ semi-sphere} \Big)$$

$$\Rightarrow \quad F_\nu = 2\pi \int_0^{\pi/2} I_\nu(\theta) \sin\theta \cos\theta \, d\theta$$

NOTE: $\nu = c/\lambda \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2} \longrightarrow F_\lambda = \frac{c}{\lambda^2} F_\nu$

$(F_\nu \, d\nu = F_\lambda \, d\lambda)$

Momentum of photon is E/c

$\rightarrow$ momentum flux along the ray at an angle θ $= \frac{dF_\nu}{c}$

$\rightarrow$ component of momentum flux normal to dA $= \frac{dF_\nu}{c} \cos\theta$

$$\Rightarrow \quad P_\nu = \int \frac{dF_\nu}{c} \cos\theta = \int \frac{I_\nu}{c} \cos^2\theta \, d\Omega$$

$dF_\nu = I_\nu \cos\theta \, d\Omega$

$$\Rightarrow \quad \boxed{P_\nu = \frac{1}{c} \int I_\nu \cos^2\theta \, d\Omega}$$

momentum flux

$[P_\nu] = \frac{\text{dynes}}{\text{cm}^2 \text{ Hz}} \leftarrow$ units of force because force $= \frac{d\,\text{momentum}}{dt}$

Integrating over frequencies:

$F = \int F_\nu \, d\nu \qquad [F] = \frac{\text{erg}}{\text{cm}^2 \text{ s}}$

$p = \int P_\nu \, d\nu \qquad [p] = \frac{\text{dynes}}{\text{cm}^2}$

$I = \int I_\nu \, d\nu \qquad [I] = \frac{\text{erg}}{\text{cm}^2 \text{ s ster}}$

SPECIFIC ENERGY DENSITY

$\boxed{u_\nu}$ = energy per unit of volume per unit of frequency range

Consider energy density per unit solid angle $u_\nu(\Omega)$

$\hookrightarrow \quad dE = u_\nu(\Omega)\, dV\, d\Omega\, d\nu$

volume element $dV = dA\, c\, dt$

cylinder of base dA & length $c\,dt$

$$\longrightarrow \quad dE = u_\nu(\Omega)\, dA\, c\, dt\, d\Omega\, d\nu \qquad (*)$$

In a time dt, all the radiation in the cylinder will pass out of it, resulting in the energy emitted by dA $\quad dE = I_\nu\, dA\, dt\, d\Omega\, d\nu \quad (**)$

Equating $(*)$ & $(**)$ $\Rightarrow \quad u_\nu(\Omega) = \dfrac{I_\nu}{c}$

Integrating over all solid angles $\quad \boxed{u_\nu = \int u_\nu(\Omega)\, d\Omega = \frac{1}{c}\int I_\nu\, d\Omega}$

$$\Leftrightarrow \quad \boxed{u_\nu = \frac{4\pi}{c} J_\nu \quad \text{where } J_\nu := \frac{1}{4\pi}\int I_\nu\, d\Omega \quad \text{MEAN INTENSITY}}$$

TOTAL RADIATION DENSITY

$$\boxed{u = \int u_\nu\, d\nu = \frac{4\pi}{c}\int J_\nu\, d\nu}$$

$$[u] = \frac{\text{erg}}{\text{cm}^3}$$

Radiation pressure of an isotropic radiation field

$$\boxed{P = \frac{1}{3} u}$$, ie, it is $\frac{1}{3}$ of the energy density.

NOTE: The intensity along rays in free space is constant, ie:

$$I_\nu = \text{constant} \Longleftrightarrow \frac{dI_\nu}{ds} = 0$$

ds ↳ differential element of length along the ray

→ see Pressure integral

RADIATIVE TRANSFER

If light passes through matter, energy may be added or subtracted by EMISSION or ABSORPTION, w/ intensity I_ν not remaining constant. "Scattering" of photons into/out of the beam can also change I_ν.

EMISSION

spontaneous emission coefficient $j = \dfrac{dE}{dV\, d\Omega\, dt}$

ie: energy emitted per unit of time per unit of solid angle & per unit of volume

Monochromatic $j_\nu = \dfrac{dE}{dV\, d\Omega\, dt\, d\nu}$

$$[j_\nu] = \frac{\text{erg}}{\text{cm}^3\ \text{s}\ \text{Hz}\ \text{str}}$$

IF isotropic emitter (ie: or randomly oriented emitters) ⇒ $j_\nu = \dfrac{1}{4\pi} P_\nu$ w/ P_ν radiated power (ie energy per unit of time) per unit volume per unit frequency.

Emissivity ϵ_ν := energy emitted spontaneously per unit frequency, per unit time, per unit mass,

$$[\epsilon_\nu] = \frac{erg}{g\ s\ Hz}$$

IF emission isotropic $\Rightarrow$ $dE = \epsilon_\nu \underbrace{\rho\, dV}_{mass}\, dt\, d\nu \frac{d\Omega}{4\pi}$

ρ mass density of emitting medium

$\frac{d\Omega}{4\pi}$: fraction of energy radiated into $d\Omega$

Comparing w/ $dE = j_\nu\, dV\, d\Omega\, dt\, d\nu$

$$\Rightarrow \boxed{j_\nu = \frac{\epsilon_\nu \rho}{4\pi} \quad \text{for isotropic emission.}}$$

In traveling a distance ds, a beam of cross section dA travels through a volume $dV = dA\, ds$. The intensity added to the beam by spontaneous emission is $dI_\nu = j_\nu\, ds$

ABSORPTION

Absorption coefficient α_ν

Loss of intensity in a beam as it travels a distance ds: $dI_\nu = -\alpha_\nu I_\nu\, ds$

NOTE: $[\alpha_\nu] = \frac{1}{cm}$ $\alpha_\nu > 0$ if energy is taken out, hence the "−" sign.

Microscopic Model: particles w/ density n (#/volume) present an effective absorbing area, or CROSS SECTION σ_ν ($[\sigma_\nu] = cm^2$). These absorbers are randomly distributed.

The energy absorbed out of the beam $-dI_\nu\, dA\, d\Omega\, dt\, d\nu = I_\nu (n \sigma_\nu\, dA\, ds)\, d\Omega\, dt\, d\nu$

$n\, dA\, ds$ = total # of absorbers

$* \sigma_\nu$ total area

$\Rightarrow dI_\nu = -n\sigma_\nu I_\nu ds \Rightarrow \boxed{\alpha_\nu = n\sigma_\nu}$

Compared w/ $dI_\nu = -\alpha_\nu I_\nu ds$

Often we write $\boxed{\alpha_\nu = \rho \kappa_\nu}$ w/ ρ mass density $\left(\frac{g}{cm^3}\right)$

$\boxed{[\kappa_\nu] = \frac{cm^2}{g}}$ is the mass absorption coefficient aka opacity coefficient

NOTE: We consider "absorption" to include BOTH true absorption & stimulated emission, because Both proportional to $I_\nu \Rightarrow$ net absorption can be positive or negative depending on whether true absorption or stimulated emission dominates.

EQUATION OF RADIATIVE TRANSFER: $\boxed{\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu}$

To solve for the intensity in an emitting & absorbing medium.

When scattering is present → more complicated, because emission into $d\Omega$ depends on I_ν in a solid angle $d\Omega'$!

$\Rightarrow$ integro-differential equation to be solved numerically.

CASE #1 EMISSION ONLY: $\alpha_\nu = 0$

$$\Rightarrow \frac{dI_\nu}{ds} = j_\nu \xrightarrow{\text{solution}} I_\nu(s) = I_\nu(s_0) + \underbrace{\int_{s_0}^{s} j_\nu(s')\,ds'}$$

the increase in brightness is equal to the emission coefficient integrated along the line of sight.

CASE #2 ABSORPTION ONLY : $j_\nu = 0$

$$\Rightarrow \frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \Rightarrow I_\nu(s) = I_\nu(s_0)\, \underbrace{e^{-\int_{s_0}^{s} \alpha_\nu(s')\,ds'}}$$

the brightness decreases along the ray by the exponential of the absorption coefficient integrated along the line of sight.

Let's define OPTICAL DEPTH τ_ν as $d\tau_\nu = \alpha_\nu ds$

$\Updownarrow$ eq.

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')\,ds'$$

Medium is:

OPTICALLY THICK or OPAQUE
if $\tau_\nu > 1$

OPTICALLY THIN or TRANSPARENT
if $\tau_\nu < 1$

the photon can traverse the medium w/o being absorbed

arbitrary; it sets the zero point for the optical depth scale.

usually measured along the path of a travelling ray; if backwards, a minus sign appears before the integral.

It depends on the medium & frequency of photons

$\boxed{\text{SOURCE FUNCTION}}$ $S_\nu := j_\nu / \alpha_\nu$

transfer eq. $\Rightarrow$ $$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

NOTE: τ_ν & S_ν are often used in place of α_ν & j_ν

Multiply by e^{τ} & defining $\mathcal{S} = S_\nu e^{\tau}$ $\quad \mathcal{I} = I_\nu e^{\tau}$

transfer eq. $\Rightarrow$ $$\frac{d\mathcal{I}}{d\tau_\nu} = \mathcal{S} \xrightarrow{\text{solution}} \mathcal{I}(\tau_\nu) = \mathcal{I}(0) + \int_0^{\tau_\nu} \mathcal{S}(\tau_\nu')\,d\tau_\nu'$$

$$\boxed{I_\nu(\tau_\nu) = I_\nu(0)\,e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')}\, S_\nu(\tau_\nu')\, d\tau_\nu'}$$ (9)

Formal solution of the transfer eq.

↑ initial intensity diminished by absorption $e^{-\tau_\nu}$

↑ integrated source diminished by absorption

EXAMPLE: S_ν = constant

$$\Rightarrow I_\nu(\tau_\nu) = I_\nu(0)\,e^{-\tau_\nu} + S_\nu\left(1 - e^{-\tau_\nu}\right) =$$

$$= S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right)$$

As $\tau_\nu \to \infty \Rightarrow I_\nu \to S_\nu$ (complication would be if scattering is present)

IF $I_\nu > S_\nu \Rightarrow \dfrac{dI_\nu}{d\tau_\nu} < 0$ & I_ν decreases along the ray

$I_\nu < S_\nu \Rightarrow I_\nu$ tends to increase along the ray

NOTE: The source function S_ν is the quantity that the specific intensity I_ν tries to approach, & does if given sufficient optical depth.

TRANSFER EQ. $\overset{\text{equiv.}}{\Longleftrightarrow}$ RELAXATION PROCESS

Mean free path of photons := average distance a photon can travel through an absorbing material w/o being absorbed.

From $I_\nu(s) = I_\nu(s_0)\, e^{-\int_{s_0}^{s} \alpha_\nu(s')\,ds'}$ (solution of transfer eq. for $j_\nu = 0$ ie, absorption only)

the probability of a photon travelling at least an optical depth τ_ν is $e^{-\tau_\nu}$ & hence $\langle \tau_\nu \rangle = 1$

$\Rightarrow$ $\boxed{\text{mean optical depth} \quad \langle \tau_\nu \rangle := \int_0^\infty \tau_\nu e^{-\tau_\nu}\, d\tau_\nu} = 1$

simple definition of mean for a continuous function

The mean physical distance traveled in a homogeneous medium is the $\boxed{\text{mean free path } l_\nu}$, determined from

$$\langle \tau_\nu \rangle = \alpha_\nu l_\nu = 1 \iff \boxed{l_\nu = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}}$$

NOTE: same definition for the local mean path in an inhomogeneous material

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If a medium absorbs radiation, the radiation exerts a force on the medium, as photons carry momentum $E/c = p$

Radiation flux vector $\vec{F}_\nu := \int I_\nu \vec{n}\, d\Omega$ $\hat{n}$ normal along direction of the ray.

The vector momentum per unit area per unit time per unit path length absorbed by the medium is $\vec{\mathcal{F}} = \frac{1}{c}\int \alpha_\nu \vec{F}_\nu\, d\nu$ Force per unit volume ($dV = dA\,ds$) imparted onto the medium by the photons.

$p = E/c$ $\quad dI_\nu = -\alpha_\nu I_\nu\, ds$ $\quad \frac{\text{momentum}}{dt\, dA\, ds} \equiv \frac{F}{dV}$

force per unit mass of material $\quad \underline{\vec{f}} = \frac{1}{c}\int \kappa_\nu \vec{F}_\nu \, d\nu$

$(\vec{f} = \vec{\mathcal{F}}/\rho)$ $\quad$ (using $\alpha_\nu = \rho \kappa_\nu$)

NOTE: $\vec{\mathcal{F}}$ & $\vec{f}$ assume α_ν is isotropic & no momentum imparted by the emission of radiation (true for isotropic emission).

THERMAL RADIATION

Thermal radiation is radiation emitted by matter in thermal equilibrium.
Blackbody radiation is radiation which is itself in thermal equilibrium.
↳ homogeneous & isotropic (it does not depend on direction)
only depends on temperature

Kirchhoff's Law for thermal emission $j_\nu = \alpha_\nu B_\nu(T)$

w/ $B_\nu(T)$ Planck's function

relation between α_ν, j_ν, & temperature of the matter T

Transfer Eq. for thermal radiation

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + B_\nu(T)$$ (see pg 8)

$I_\nu = B_\nu$ Blackbody radiation
$S_\nu = B_\nu$ Thermal radiation

Thermal radiation becomes blackbody radiation only for optically thick media.

Photons are massless bosons, so the Einstein-Bose statistics applies

$$n(\varepsilon) = \frac{g(\varepsilon)}{e^{\alpha+\varepsilon/KT} - 1}$$

Photons → $\alpha = 0$ (mass-less bosons)

$$\Rightarrow n(\epsilon) = \frac{g(\epsilon)}{e^{\epsilon/KT} - 1}$$

$g(\epsilon)$ → # of possible particles states of energy ϵ

$n(\epsilon)$ → number density of gas particles of energy ϵ

Total number density $n := \int n(\epsilon)\, d\epsilon$

$$g(p)\,dp = \frac{2}{h^3} 4\pi p^2\, dp$$

(in momentum space)

$$P_\nu = \frac{h\nu}{c} \qquad E = h\nu$$

Energy density $u(\nu)\,d\nu = E(\nu)\,n(\nu)\,d\nu = h\nu\, n(\nu)\,d\nu$

$$n(\nu)\,d\nu = \frac{g(\nu)}{e^{h\nu/KT}-1}\,d\nu = \frac{8\pi}{c^3}\nu^2 d\nu \frac{1}{e^{h\nu/KT}-1}$$

$E = h\nu$

$$g(\nu)\,d\nu = g(p)\,dp = \frac{2}{h^3}4\pi p^2 dp = \frac{8\pi}{c^3}\nu^2 d\nu$$

$p = h\nu/c$

$dp = \frac{h}{c}d\nu$

$$\boxed{u(\nu)\,d\nu = \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/KT}-1}}$$

$[u(\nu)] = \frac{erg}{cm^3\ Hz}$

$$\boxed{u(\lambda)\,d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda KT}-1}}$$

ENERGY DENSITY for Black body radiation (e.g. cosmic microwave background rad.)

using $\nu = c/\lambda$

$d\nu = \frac{c}{\lambda^2}d\lambda$

Total energy $u := \int_0^\infty u(\nu)\,d\nu = \int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/KT}-1} =$

$x = h\nu/KT$

$\nu = \frac{KTx}{h}$

$d\nu = \frac{KT}{h}dx$

$$= \frac{8\pi h}{c^3}\left(\frac{KT}{h}\right)^4 \underbrace{\int_0^\infty \frac{x^3 dx}{e^x-1}}_{\pi^4/15}$$

$$\boxed{u = \frac{8\pi^5 K^4}{15c^3h^3}T^4 = aT^4 \quad \text{w/} \quad a = 7.565\times 10^{-15}\ \frac{erg}{cm^3}\frac{1}{{}^\circ K^4}}$$

$[u] = \frac{erg}{cm^3}$

Stefan-Boltzmann law for blackbody radiation

For isotropic radiation $u = \frac{4\pi}{c}\int B_\nu(T)\,d\nu$ (see pg 4) 14

ie: $u(\nu) \equiv u_\nu = \frac{4\pi}{c} \underbrace{B_\nu(T)}_{\text{Planck's function}}$

$$\boxed{\begin{aligned} B_\nu(T) &= \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/KT}-1} \\ B_\lambda(T) &= \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda KT}-1} \end{aligned}}$$ Planck law → Fig

$$[B_\nu(T)] = \frac{erg}{s\ cm^2\ Hz\ str} \qquad [B_\lambda(T)] = \frac{erg}{s\ cm^2\ \mathring{A}\ st}$$

The emergent density flux from an isotropically emitting surface (ie, a black body's surface) is, from pg 2+3

$$F_\nu = 2\pi\int_0^{\pi/2} B_\nu(T)\sin\theta\cos\theta\,d\theta = 2\pi B_\nu(T)\underbrace{\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta}_{\frac{1}{2}\left[\sin^2\theta\right]_0^{\pi/2} = \frac{1}{2}}$$

$$= \pi B_\nu(T)$$

$$\boxed{\begin{aligned} F_\nu &= \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{h\nu/KT}-1} \\ F_\lambda &= \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda KT}-1} \end{aligned}} \qquad [F_\nu(T)] = \frac{erg}{s\ cm^2\ Hz}$$

Emergent flux (total power radiated per unit of area) from an isotropically emitting surface is

$$F = \int_0^\infty F_\lambda \, d\lambda = \int_0^\infty \frac{2\pi h c^2}{\lambda^5} \frac{1}{e^{hc/\lambda KT} - 1} = \frac{2\pi^5}{15 c^2} \frac{K^4}{h^3} T^4$$

$$x = hc/\lambda KT$$

$$\int_0^\infty \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

$$\Rightarrow \boxed{F = \sigma T^4}$$

$$\sigma = \frac{2\pi^5 K^4}{15 c^2 h^3} = 5.67 \times 10^{-5} \frac{\text{erg}}{\text{s cm}^2 \, {}^\circ K^4}$$

Stefan-Boltzmann law from surface Blackbody

~~Rayleigh~~

IF $h\nu \ll KT \rightarrow e^{h\nu/KT} - 1 = \frac{h\nu}{KT} + \dots$

$$\Rightarrow \boxed{B_\nu^{RJ}(T) = \frac{2\nu^2}{c^2} KT}$$

RAYLEIGH-JEANS LAW

NOTE: It applies at low energies/frequencies (eg; in the radio, it almost always applies). It is the "straight-line" part in the $\log B_\nu - \log \nu$ plot.

If the RJ law applied to all ν, the total amount of energy $\propto \int \nu^2 d\nu = \infty \Rightarrow$ ULTRAVIOLET CATASTROPHE

For $h\nu \gg KT$, the discrete quantum nature of photons must be taken into account.

IF $h\nu \gg KT$ $\longrightarrow$ $e^{h\nu/KT} - 1 \approx e^{h\nu/KT}$

$$\Rightarrow \boxed{B_\nu^W(T) = \frac{2h\nu^3}{c^3} e^{-h\nu/KT}}$$

WIEN'S LAW

ie: the brightness decreases very rapidly w/ frequency once the maximum is reached

NOTE: of two blackbody curves, the one w/ higher T lies entirely above the other

$$\frac{\partial B_\nu(T)}{\partial T} = \frac{2h^2\nu^4}{c^2 K T^2} \frac{e^{h\nu/KT}}{\left(e^{h\nu/KT} - 1\right)^2} > 0$$

$\Rightarrow$ @ any frequency, if T increases, also $B_\nu(T)$ increases

Also $B_\nu \to 0$ as $T \to 0$

$B_\nu \to \infty$ as $T \to \infty$

NOTE: Solving $\left.\frac{\partial B_\nu}{\partial \nu}\right|_{\nu = \nu_{max}} = 0$ $\longrightarrow$ ν_{max} frequency of the peak of $B_\nu(T)$

$\Updownarrow$ equivalent to set $x := h\nu_{max}/KT$ & solve

$x = 3(1 - e^{-x})$ (numerically)

$$\Rightarrow \boxed{\begin{aligned} h\nu_{max} &= 2.82\, KT \\ \frac{\nu_{max}}{T} &= 5.88 \times 10^{10} \ \frac{\text{Hz}}{{}^\circ\text{K}} \end{aligned}}$$

Wien's displacement law

ie: the peak frequency of the BB law shifts linearly w/ T.

Doing similarly w/ $\left.\frac{\partial B_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_{max}} = 0$ using $y = hc/\lambda_{max}KT$

$\Leftrightarrow y = 5(1-e^{-y})$

$= 4.97$

$$\Rightarrow \boxed{\lambda_{max} T = 0.290 \text{ cm °K}}$$

NOTE: The peaks of B_ν & B_λ do not occure at the same places in wavelength or frequency, ie:

$\lambda_{max} \nu_{max} \neq c$

NOTE: IF $\nu \ll \nu_{max} \rightarrow$ RJ Law
IF $\nu \gg \nu_{max} \rightarrow$ Wien law

BRIGHTNESS temperature $\boxed{T_b}$: temperature of the blackbody having the same brightness I_ν at a given frequency, ie $I_\nu = B_\nu(T_b)$

T_b is used especially in radio astronomy, where the Rayleigh-Jeans law is applicable $\Rightarrow I_\nu = \frac{2\nu^2}{c^2} K T_b$

$$\Rightarrow \boxed{T_b = \frac{c^2}{2\nu^2 K} I_\nu} \quad (\text{for } h\nu \ll KT)$$

The transfer eq. for thermal emission becames: $\frac{dT_b}{d\tau_\nu} = -T_b + T$
w/ T temperature of the material.

IF T = const $\Rightarrow T_b = T_b(0)e^{-\tau_\nu} + T(1-e^{-\tau_\nu})$, $h\nu \ll KT$

IF $\tau_\nu \gg$ $\rightarrow T_b \rightarrow T$ brightness Temperature of radiation approaches T material

NOTE: $T_b(\nu)$ is a function of frequency. Only for a perfect blackbody source, T_b = constant.

COLOR TEMPERATURE $\boxed{T_c}$

This is the temperature of a blackbody having the same SHAPE of the continuum spectrum.

$$\Rightarrow \quad \frac{F_{\lambda_1}}{F_{\lambda_2}} \quad \text{or} \quad \text{Wien's displacement law} \quad T = \frac{0.290 \text{ cm } °K}{\lambda_{max}}$$

$$m_B = -2.5 \log F_{\lambda_B}$$
$$m_V = -2.5 \log F_{\lambda_V}$$

$$m_B - m_V = -2.5 \log \frac{F_{\lambda_B}}{F_{\lambda_V}} = 2.5 \log \frac{F_{\lambda_V}}{F_{\lambda_B}}$$

$$= \alpha + \frac{\beta}{T_c} + f(T_c)$$

function that varies slowly w/ T_c

EFFECTIVE TEMPERATURE $\boxed{T_{eff}}$

This is the temperature of the blackbody with the same radiated power per unit of area F.

$$\Rightarrow \quad F = \sigma T_{eff}^4 \quad \Rightarrow \quad \boxed{T_{eff} = \sqrt[4]{F/\sigma}}$$

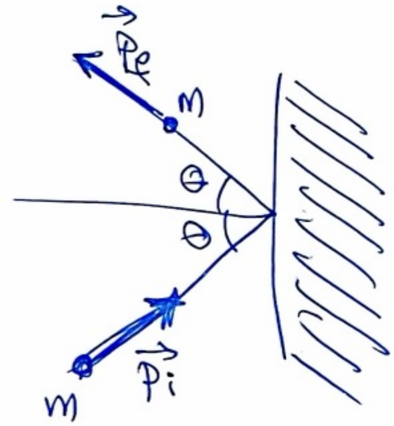
NOTE: T_{eff} & T_b depend on the magnitude of the source intensity (ie, the absolute vertical scale).
T_c depends only on the shape of the observed spectrum (ie: I don't need to know the distance of the source)

THE PRESSURE INTEGRAL

The microscopic source of pressure in a perfect gas is particle bombardment $\Rightarrow$ transfer of momentum, hence a force ($F = dp/dt$). The average force per unit of area is the pressure ($P = F/A$)

In thermal equilibrium, the angular distribution of particle momenta is isotropic, ie: particles are moving w/ equal probability in all directions.

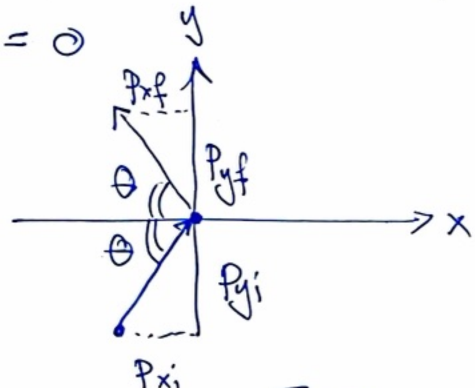
Variation of the momentum of the particle m:



$$\Delta p_x = p_{xf} - p_{xi} = -2p_{xi}$$

since the final component $p_{xf} = -p_{xi}$, ie: it changes sign

$\Delta p_y = 0$



$$\text{Pressure} \equiv P := \frac{\text{Force}}{\text{Area}} \qquad \text{Force} := \frac{\Delta p}{\Delta t}$$

momentum

$$\rightarrow P = \frac{\Delta p}{\Delta A \, \Delta t}$$

Momentum transfered to the surface is $-\Delta p_x$

We can also write $\Delta p = 2p\cos\theta$ momentum transfered to the surface

PI2

Let $F(\theta, p)\, d\theta\, dp :=$ # particles w/ momentum p in the range dp striking the surface per unit of area per unit of time from all directions inclined at an angle θ to the normal in the range $d\theta$

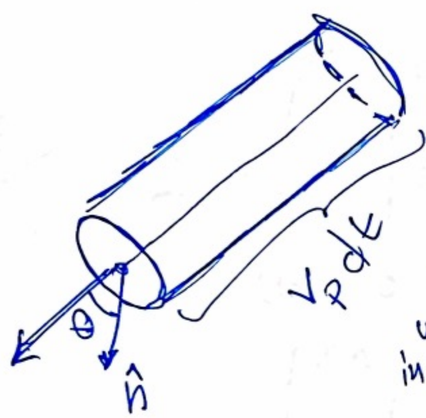
The contribution to pressure from these particles

$$dP := \frac{\Delta p}{\Delta t\, \Delta \text{area}} = 2p\cos\theta\, F(\theta, p)\, d\theta\, dp$$

$$F(\theta, p)\, d\theta\, dp = \underbrace{v_p \cos\theta}\ n(\theta, p)\, d\theta\, dp$$

volume of such particles capable of passing through the unit surface in unit of time

density of particles w/ momentum p in the range dp moving in the cone of directions inclined at an angle θ in the range $d\theta$



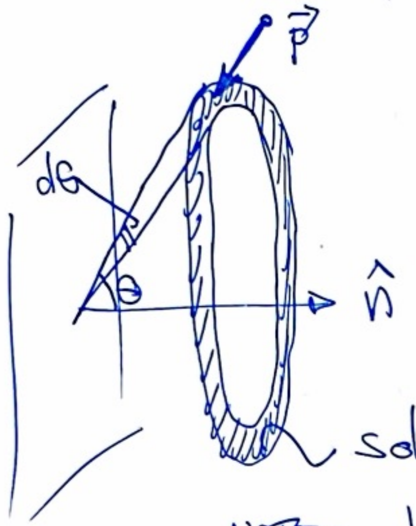
$$\hookrightarrow \underset{\text{volume}}{V} = v_p\, dt \cdot dA \cdot \cos\theta$$

$$\frac{V}{dA\, dt} = v_p \cos\theta$$

$\underbrace{\qquad\qquad\qquad}$

density of particles moving in the prescribed cone

Because of ISOTROPY $\Rightarrow n(\theta, p)\, d\theta\, dp = \frac{2\pi \sin\theta\, d\theta}{4\pi} \underbrace{n(p)dp}_{\text{total number density of particles of momentum } p \text{ in } dp}$ PI3



NOTE: $d\Omega = \sin\theta\, d\theta\, d\phi$

w/ $\phi \in [0, 2\pi]$

$\theta \in [0, \pi/2]$ semi-sphere

$$\Rightarrow d\Omega = 2\pi \sin\theta\, d\theta$$

$$\Rightarrow P = \int_{\theta=0}^{\pi/2} \int_0^{\infty} 2p \cos\theta\, V_p \cos\theta \frac{\cancel{2\pi} \sin\theta}{\cancel{4\pi}} d\theta\, n(p)\, dp =$$

$$= \int_0^{\pi/2} \int_0^{\infty} p\, V_p\, n(p) \cos^2\theta \sin\theta\, d\theta\, dp =$$

$$= \int_0^{\infty} p\, V_p\, n(p)\, dp \underbrace{\int_0^{\pi/2} \cos^2\theta\, d(\cos\theta)}_{1/3}$$

$$\Rightarrow P = \frac{1}{3} \int_0^{\infty} p\, V_p\, n(p)\, dp$$ PRESSURE Integral for a perfect isotropic gas.

NOTE: The relation between p & V_p depends upon relativistic considerations, whereas $n(p)$ depends on the type of particles & quantum statistics.

For an isotropic radiation flux (PI 4)

$$P = \frac{1}{3}\int_0^{\infty} p\, v_p\, n(p)\, dp = \frac{1}{3}\int_0^{\infty} \underset{\overset{\uparrow}{p}}{\frac{h\nu}{c}}\ \underset{\overset{\uparrow}{v_p}}{c}\ n(\nu)\, d\nu =$$

$$= \underbrace{\frac{1}{3}\int_0^{\infty} h\nu\, n(\nu)\, d\nu}$$

:= energy density of photons u

$$\Longrightarrow P = \frac{1}{3} u = \frac{1}{3} a T^4$$

w/ $a = 7.565 \times 10^{-15} \frac{\text{erg}}{\text{cm}^3\, {}^\circ\text{K}^4}$

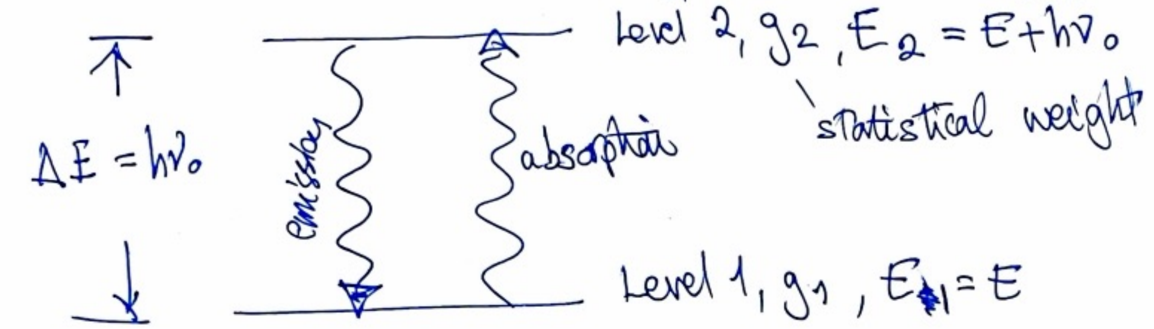
EINSTEIN COEFFICIENTS

Kirchhoff's law $j_\nu = \alpha_\nu B_\nu$ relates emission to absorption for a thermal emitter

⇓

there must be some relation between emission & absorption at the microscopic level

Consider two discrete energy levels (see fig):

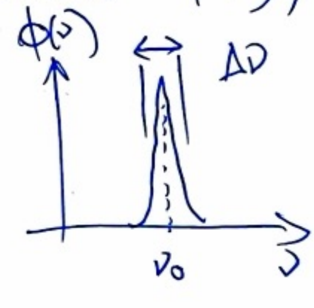


3 processes:

1) SPONTANEOUS EMISSION when the system is in level 2 & drops to level 1 by emitting a photon, occurring even in the absence of a radiation field

A_{21} = probability of transition for spontaneous emission per unit of time

2) ABSORPTION which occurs in the presence of photons of energy $h\nu_0$. Since no self-interaction of the radiation field, the probability of absorption is $\propto$ to the density of photons (or the mean intensity) @ frequency ν_0. Note that the energy difference between the two levels is not infinitely sharp but described by a line profile function $\phi(\nu)$, sharply peaked @ $\nu = \nu_0$ & normalized so that $\int_0^\infty \phi(\nu)\, d\nu = 1$



$\phi(\nu) \approx$ effectiveness of frequencies around ν_0 causing transition

$\Rightarrow$ $B_{12}\bar{J}$ = transition probability for absorption per unit of time

w/ $\bar{J} \equiv \int_0^\infty J_\nu \, \phi(\nu) \, d\nu$

* B_{12}: Einstein B-coefficient

3) STIMULATED EMISSION

$B_{21}\bar{J}$ = transition probability for stimulated emission per unit of time

NOTE: When J_ν slowly changing over the interval $\nu_0 \pm \frac{\Delta\nu}{2}$

$\Rightarrow$ $\phi(\nu) \simeq \delta$-function $\Rightarrow$ $B_{12} J_{\nu_0}$

$B_{21} J_{\nu_0}$

NOTE: Energy density u_ν is often used instead of J_ν to define Einstein B-coefficients $\Rightarrow$ $c/4\pi$ differing definitions

In thermodynamic equilibrium

$\Downarrow$

\# transitions per unit of time per unit of volume into level 1 $\equiv$ \# transitions .... into level 2

If n_1, n_2 number densities of atoms in levels 1 & 2

$$\Rightarrow \underbrace{n_1 B_{12} \bar{J}}_{\text{absorption}} = \underbrace{n_2 A_{21} + n_2 B_{21} \bar{J}}_{\text{emission}}$$

Solving for $\bar{J}$ $\Rightarrow$ $$\bar{J} = \frac{A_{21}/B_{21}}{\frac{n_1}{n_2}\frac{B_{12}}{B_{21}} - 1}$$

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In thermodynamic equilibrium, Boltzmann's eq:

$$\frac{n_1}{n_2} = \frac{g_1 e^{-E/KT}}{g_2 e^{-(E+h\nu_0)/KT}} = \frac{g_1}{g_2} e^{h\nu_0/KT}$$

$$\Rightarrow \quad \bar{J} = \frac{A_{21}/B_{21}}{\frac{g_1 B_{12}}{g_2 B_{21}} e^{h\nu_0/KT} - 1} \quad (*)$$

($\square$)

But in TE, $J_\nu = B_\nu$ Planck's law & $\bar{J} = B_\nu$ as B_ν varies slowly on the scale of $\Delta\nu$.

Comparing $(*)$ & ($\square$)

$\Rightarrow$ Einstein relations

$$\boxed{\begin{aligned} g_1 B_{12} &= g_2 B_{21} \\ A_{21} &= \frac{2h\nu^3}{c^2} B_{21} \end{aligned}}$$

NOTE: Since there is no reference to temperature, Einstein relations hold also when not in thermodynamic eq. Knowing one of the Einstein coefficients, allows to derive the other two.

Assuming that the spontaneous emission radiation has the same line profile function $\phi(\nu)$ as absorption

$$\Rightarrow \quad j_\nu \, dV \, d\Omega \, d\nu \, dt = \frac{h\nu_0}{4\pi} \phi(\nu) \, n_2 \, A_{21} \, dV \, d\Omega \, d\nu \, dt$$

energy emitted in volume dV, solid angle $d\Omega$, frequency range $d\nu$, & time dt

contribution by each atom of energy over 4π solid angle for each transition

density of atom

probability of transition per unit of time

$$\boxed{j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu)}$$ emission coefficient expressed in terms of A_{21}

For absorption, the total energy absorbed in time dt in volume dV is $dV\,dt\,\frac{h\nu_0}{4\pi} n_1 B_{12} \int d\Omega \int I_\nu \phi(\nu)\,d\nu$

(from definition of $B_{12}\bar{J}$)

$\Rightarrow$ energy absorbed out of a beam in frequency range $d\nu$ solid angle $d\Omega$ time dt in volume dV is $= dV\,dt\,d\Omega\,d\nu\,\frac{h\nu_0}{4\pi} n_1 B_{12}\phi(\nu) I_\nu$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

Using $dV = ds\,dA \qquad dE = dI_\nu\,dA\,dt\,d\Omega\,d\nu$

$$\Rightarrow \quad dI_\nu \cancel{dA}\,\cancel{dt}\,\cancel{d\Omega}\,\cancel{d\nu} = \underbrace{dV}_{ds\,\cancel{dA}}\,\cancel{dt}\,\cancel{d\Omega}\,\cancel{d\nu}\,\frac{h\nu_0}{4\pi} n_1 B_{12} \phi(\nu) I_\nu$$

Comparing w/ $dI_\nu := -\alpha_\nu I_\nu\,ds$

$$\Rightarrow \quad \alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12}\,\phi(\nu)$$ absorption coefficient uncorrected for stimulated emission

We can treat stimulated emission as negative absorption

$$\Rightarrow \quad \boxed{\alpha_\nu = \frac{h\nu}{4\pi}\phi_\nu \left(n_1 B_{12} - n_2 B_{21}\right)}$$

ABSORPTION coefficient

$\Rightarrow$ Transfer equation using Einstein coefficients: 23

$$\frac{dI_\nu}{ds} = -\frac{h\nu}{4\pi}\left(n_1 B_{12} - n_2 B_{21}\right)\phi(\nu) I_\nu + \frac{h\nu}{4\pi} n_2 A_{21}\phi(\nu)$$

Source function $S_\nu := \dfrac{j_\nu}{\alpha_\nu} = \dfrac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$

Using Einstein relations

$B_{21}/B_{12} = g_1/g_2$

$$\boxed{\begin{aligned} \alpha_\nu &= \frac{h\nu}{4\pi}\phi(\nu)\, n_1 B_{12}\left(1 - \frac{g_1 n_2}{g_2 n_1}\right) \\ S_\nu &= \frac{2h\nu^3}{c^2}\left(\frac{g_2 n_1}{g_1 n_2} - 1\right)^{-1} \end{aligned}}$$

GENERALIZED Kirchhoff's law

$A_2 = \dfrac{2h\nu^3}{c^2} B_{21}$

$B_{21}/B_{12} = g_1/g_2$

Three cases:

① THERMAL EMISSION (LTE)

If matter in thermal equilibrium w/ itself $\rightarrow \dfrac{n_1}{n_2} = \dfrac{g_1}{g_2} e^{h\nu/kT}$

ie: matter in LTE, local thermodynamic equilibrium

$$\Rightarrow \quad \alpha_\nu = \frac{h\nu}{4\pi} n_1 B_{12}\left(1 - e^{-h\nu/kT}\right)\phi(\nu)$$

$$S_\nu = B_\nu(T) \qquad \text{(Kirchhoff's law)}$$

② NON THERMAL EMISSION, when $\dfrac{n_1}{n_2} \neq \dfrac{g_1}{g_2} e^{h\nu/kT}$

eg: for a plasma if radiating particles did not have a Maxwellian velocity distribution, or if the atomic population did not obey the Maxwell-Boltzmann distribution law.

③ INVERTED POPULATION, MASERS

For a system in thermal equilibrium $\frac{n_2 g_1}{n_1 g_2} = e^{-h\nu/kT} < 1$

$\Rightarrow \frac{n_1}{g_1} > \frac{n_2}{g_2}$ generally satisfied even when the material is not in thermal equilib.

$\Rightarrow$ NORMAL POPULATIONS

It is however possible to put enough atoms in the upper state so that we have INVERTED POPULATIONS $\frac{n_1}{g_1} < \frac{n_2}{g_2}$

$\Rightarrow \alpha_\nu < 0$ ie: rather than decreases along the ray, the intensity increases.

This system is a MASER (microwave amplification by stimulated emission of radiation) or LASER (for light)

SCATTERING EFFECTS / RANDOM WALKS

Thermal emission: amount of radiation emitted by an element is not dependent on the radiation field incident.

Scattering: depends completely on the amount of radiation falling on the element

ISOTROPIC scattering: the scattered radiation is emitted equally into equal solid angles (emission coefficient independent of direction)

COHERENT scattering: total amount of radiation emitted per unit frequency range is equal to the total amount absorbed in same ν range.
(ie, scattering from non-relativ. e's)

Emission coefficient for coherent isotropic scattering

$$\underbrace{j_\nu = \sigma_\nu}_{} J_\nu$$

absorption coefficient of scattering process, aka scattering coefficient

$$S_\nu = \frac{j_\nu}{\sigma_\nu} = J_\nu = \underbrace{\frac{1}{4\pi}\int I_\nu \, d\Omega}_{\text{mean intensity w/in the emitting material}}$$

source function for scattering

$\Rightarrow$ Transfer equation for pure scattering

$$\frac{dI_\nu}{ds} = -\sigma_\nu \left(\underbrace{I_\nu}_{\text{absorbed}} - \overbrace{J_\nu}^{\text{re-emission}}\right)$$

integro - differential equation
(J_ν, source function, is not known a priori, & depends on solution I_ν)

$$\left(\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu\right)$$

Consider a photon emitted in an infinite, homogenous scattering region. The net displacement $\vec{R} = \vec{r}_1 + \vec{r}_2 + \ldots + \vec{r}_N$

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$|\vec{R}| = 0$ because the average displacement will be zero

Mean square displacement

$$\ell_*^2 \equiv \langle \vec{R}^2 \rangle = \langle \vec{r}_1^{\,2} \rangle + \langle \vec{r}_2^{\,2} \rangle + \ldots + \langle \vec{r}_N^{\,2} \rangle + \underbrace{\{ + 2\langle \vec{r}_1 \cdot \vec{r}_2 \rangle + 2 \langle \vec{r}_1 \cdot \vec{r}_3 \rangle + \ldots \}}_{\equiv 0} = N \ell^2$$

$$\Rightarrow \ell_*^2 = N\ell^2 \longrightarrow \boxed{\ell_* = \sqrt{N}\,\ell}$$

ℓ — mean free path of photon

N — # of scatterings

ℓ_* — root mean square net displacement of the photon

For regions of large enough optical depth

$$\sqrt{N} = \frac{\ell_*}{\ell} \simeq \frac{L}{\ell} = L\sigma n = \tau$$

τ — optical thickness of the medium

L typical size of the medium

$\ell = \frac{1}{\sigma n}$ mean free path

$$\Rightarrow \boxed{N = \tau^2 \quad \text{for } \tau \gg 1}$$

# of scatterings of the photon before it escapes.

For regions of small optical thickness, the mean number of scattering is small, of order $1 - e^{-\tau} \approx \tau$

$$\Rightarrow \boxed{N \simeq \tau \quad \text{for } \tau \ll 1}$$

(the majority of photons escape w/o interacting)

optically thin

Case of material w/ absorption coefficient α_ν for thermal emission & a scattering coefficient σ_ν for the coherent isotropic scattering 27

$\Rightarrow$ Transfer eq: $\frac{dI_\nu}{ds} = -\alpha_\nu(I_\nu - B_\nu) - \sigma_\nu(I_\nu - J_\nu)$

$$= -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu)$$

w/ source function $S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu}$

ie: average of the two separate source functions weighted by their respective absorption coefficients

$\rightarrow$ $\boxed{\alpha_\nu + \sigma_\nu \quad \text{Net absorption coefficient}}$ aka extinction coefficient

$d\tau_\nu = (\alpha_\nu + \sigma_\nu)\, ds$ optical depth

Two extreme situations:

1) matter element is deep inside a medium at some constant T $\longrightarrow$ radiation field near thermodynamic value $J_\nu = B_\nu(T)$

$\Rightarrow S_\nu = B_\nu(T)$

2) matter element isolated in free space (ie $J_\nu = 0$)

$\Rightarrow S_\nu = \frac{\alpha_\nu B_\nu}{\alpha_\nu + \sigma_\nu}$

In general, S_ν is not known a priori ...

Applying the random walk arguments to the case of combined scattering & absorption $\quad l_\nu = \frac{1}{\alpha_\nu + \sigma_\nu}$

mean free path of photon is the inverse of the total extinction coefficient

During the random walk, the probability that a free path will end w/ a true absorption event is 28

$$\epsilon_\nu = \frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu}$$

Probability of scattering $1-\epsilon_\nu = \dfrac{\sigma_\nu}{\alpha_\nu+\sigma_\nu}$ SINGLE scattering Albedo

$\Rightarrow$ $S_\nu = (1-\epsilon_\nu)J_\nu + \epsilon_\nu B_\nu$ source function.

Consider infinite homogeneous medium.
A random walk starts w/ the thermal emission of a photon (creation) & ends, after a number of scatterings, with a true absorption (destruction).
The walk can be terminated w/ probability ϵ at the end of each free path $\rightarrow$ mean # of free paths $N = 1/\epsilon$

From $l_* = \sqrt{N}\, l \longrightarrow l_*^2 = l^2/\epsilon \rightarrow l_* = l/\sqrt{\epsilon}$

Using $l = \dfrac{1}{\alpha_\nu + \sigma_\nu}$ & $\epsilon = \dfrac{\alpha_\nu}{\alpha_\nu+\sigma_\nu}$ into l_*

$\Rightarrow$ $\boxed{l_* \cong \dfrac{1}{\sqrt{\alpha_\nu(\alpha_\nu+\sigma_\nu)}}}$ * $\Longleftrightarrow$ l_* is measurement of the net displacement between the points of creation & destruction of a typical photon: diffusion length, thermalization length, effective mean path

NOTE: l_* is generally a function of ν.

Finite medium: behavior depends on the size L of the medium, & whether it is larger or smaller than ℓ_*.

→ effective optical thickness $\tau_* = L/\ell_*$

Defining $\tau_a := \alpha_\nu L$ absorption optical thickness
$\tau_s := \sigma_\nu L$ scattering " "

From ✱ ⇒ $\tau_* \simeq \sqrt{\tau_a(\tau_a + \tau_s)}$

When $\ell_* > L$ ⇒ $\tau_* \ll 1$ ⇒ medium is effectively thin or translucent

most photons will escape by random walking out of the medium before being destroyed by a true absorption.

⇐ Monochromatic luminosity will be equal to the radiation created by thermal emission in the medium

$$\boxed{\mathcal{L}_\nu = 4\pi \alpha_\nu B_\nu V}$$

emitted power per unit frequency

Volume of medium

When $\ell_* < L$ ⇒ $\tau_* \gg 1$ ⇒ medium is effectively thick

most photons thermally emitted @ depths larger than the effective path length ℓ_* will be destroyed by absorption before leaving.

$I_\nu \to B_\nu$
$S_\nu \to B_\nu$ ⇐ conditions for the radiation to come into thermal equilibrium w/ matter

Note: in this case, $\ell^* \equiv$ distance over which thermal equilibrium of the radiation is established
→ ℓ^* thermalization length

⇒ monochromatic luminosity $\epsilon_\nu = \alpha_\nu / (\alpha_\nu + \sigma_\nu)$ $\quad \ell_* = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu+\sigma_\nu)}}$

$$\mathcal{L}_\nu \simeq 4\pi \alpha_\nu B_\nu \underbrace{A \ell_*}_{\text{effective emitting volume}} \simeq 4\pi \sqrt{\epsilon_\nu} B_\nu A$$

$\alpha_\nu \ell_* = \sqrt{\epsilon_\nu}$

estimate

$\mathcal{L}_\nu$ depends on ϵ_ν & on geometry in complex way

hence $4\pi \longleftrightarrow \pi$

only photons emitted w/in an effective path length of the boundary have a reasonable chance of escaping before being absorbed

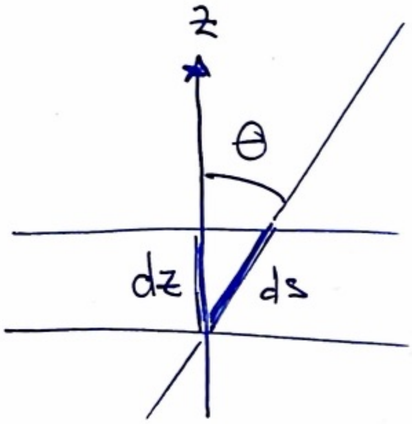
⇕

If no scattering $\epsilon_\nu \to 1$ ⇒ emission of blackbody w/ $\mathcal{L}_\nu = \pi B_\nu A$

RADIATIVE DIFFUSION

S_ν approaches B_ν @ large effective optical depths in a homogeneous medium.

Real media seldomly homogeneous, but often have high degree of local homogeneity (eg, interior of stars).



$$ds = \frac{dz}{\cos\theta} = \frac{dz}{\mu}$$

$$\mu := \cos\theta$$

Assumption: the material properties $(T, \alpha_\nu, \ldots)$ depend only on depth in the medium

plane-parallel assumption $\Leftarrow$

ie: intensity depends only on θ direction of the ray wrt normal to the planes of constant properties.

$\Rightarrow$ Transfer eq. $\dfrac{dI_\nu}{ds} = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu)$

$$\Leftrightarrow \quad \mu \frac{\partial I_\nu(z,\mu)}{\partial z} = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu)$$

Rewriting $\quad I_\nu(z,\mu) = S_\nu - \dfrac{\mu}{\alpha_\nu + \sigma_\nu}\dfrac{\partial I_\nu}{\partial z}$

When the point deep in the material $\Rightarrow$ intensity changes rather slowly on the scale of a mean free path

ie $\dfrac{\partial I_\nu}{\partial z}$ small

$\Rightarrow$ zeroth approximation $\quad I_\nu^{(0)}(z,\mu) \simeq S_\nu^{0}(T)$

Since independent on μ $\Rightarrow$ zeroth-order mean intensity $\quad J_\nu^{(0)} = S_\nu^{(0)}$ $\quad \Rightarrow I_\nu^{(0)} = S_\nu^{(0)} = B_\nu$

from $S_\nu = \dfrac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu}$

Replacing, we get a better first order approximation using $I_\nu^{(0)} = B_\nu$

$$\Rightarrow \quad I_\nu^{(1)}(z,\mu) \simeq B_\nu(T) - \frac{\mu}{\alpha_\nu + \sigma_\nu} \frac{\partial B_\nu}{\partial z}$$

Monochromatic Flux

$$F_\nu(z) = \int I_\nu^{(1)}(z,\mu) \cos\theta \, d\Omega = 2\pi \int_{-1}^{1} I_\nu^{(1)}(z,\mu)\, \mu \, d\mu$$

$d\Omega = 2\pi \sin\theta d\theta \quad \theta \in [0,\pi]$

$d\mu = \sin\theta d\theta$

The angle-independent part of $I_\nu^{(1)}$, ie $B_\nu(T)$, does not contribute to the flux

$$\Rightarrow F_\nu(z) = -\frac{2\pi}{\alpha_\nu + \sigma_\nu} \frac{\partial B_\nu}{\partial z} \int_{-1}^{1} \mu^2 d\mu =$$

$$= -\frac{4\pi}{3(\alpha_\nu + \sigma_\nu)} \frac{\partial B_\nu(T)}{\partial z} =$$

$$= -\frac{4\pi}{3(\alpha_\nu + \sigma_\nu)} \frac{\partial B_\nu(T)}{\partial T} \frac{\partial T}{\partial z}$$

Total flux $F(z) = \int_0^\infty F_\nu(z) d\nu = -\frac{4\pi}{3} \frac{\partial T}{\partial z} \int_0^\infty (\alpha_\nu + \sigma_\nu)^{-1} \frac{\partial B_\nu}{\partial T} d\nu$

Using $\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \frac{\partial}{\partial T} \int_0^\infty B_\nu d\nu = \frac{\partial B(T)}{\partial T} = \frac{4\sigma T^3}{\pi}$

$B(T) = \frac{a c}{4\pi} T^4 = \frac{\sigma}{\pi} T^4$

$F = \pi B(T) = \sigma T^4$

σ Stefan-Boltzman constant

Let's define Rosseland mean absorption coefficient α_R

$$\frac{1}{\alpha_R} := \frac{\int_0^\infty (\alpha_\nu + \sigma_\nu)^{-1} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}$$

weighted average of $\frac{1}{\alpha_\nu + \sigma_\nu}$ so that frequencies @ which extinction coeff. is small (transparent regions) tend to dominate the average.

Weighting function $\partial B_\nu / \partial T$ similar shape of Planck's function, but peaks @ $3.8 \frac{h\nu}{kT}$ instead of $2.8 h\nu/kT$

$$\Rightarrow \boxed{F(z) = -\frac{16 \sigma T^3}{3 \alpha_R} \frac{\partial T}{\partial z}}$$

Rosseland approximation of energy flux

Eq. of Radiative diffusion

NOTE: 1) the radiative energy transport deep in a star is of the same nature as a heat conduction, w/ effective heat conductivity $= \frac{16 \sigma T^3}{3 \alpha_R}$

2) The energy flux depends on only one property of the absorption coefficient, ie its Rosseland mean.

Result is general: the vector flux is in the direction opposite to the temperature gradient

Necessary assumption: all quantities change slowly on the scale of any radiation mean free path (ie: LTE)