POLYTROPES

1. $\frac{dM_r}{dr} = 4\pi r^2 \rho(r)$ EQ. of CONSERVATION OF MASS

2. $\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$ EQ. of HYDROSTATIC EQ.

3. $P = P(\rho, T, \mu)$ EQ. of STATE → to remove one variable
   (μ: chemical composition)

BUT they above do not allow to determine the structure of a star, since the required P is obtainable from an unlimited number of combinations of ρ & T. P also depends on composition (through μ).

However 1 & 2 could be solved simultaneously if $P = f(\rho)$, ie: the pressure is a function of only density.

$\Rightarrow \boxed{P = K\rho^{n+1/n}}$ w/ n polytropic index

Stellar models in hydrostatic equilibrium in which $P = K\rho^{n+1/n}$ are called polytropes.

1ST EXAMPLE

Let's consider a star that is in a state of ADIABATIC CONVECTIVE EQUILIBRIUM ("boiling"): convective equilibrium ↔ any mass element after rising & falling finds itself at the same temperature and density of the surrounding; adiabatic ↔ if the convective cells move w/o heat exchange. If P_{rad} not important $\Rightarrow P = K\rho^{\gamma}$, ie: a polytrope

w/ $\gamma = 5/3$ for an ideal monatomic gas,

w/ $n = 3/2$.

2ND EXAMPLE

Star in which P_{rad} is important

$\Rightarrow P_g = \frac{K\rho T}{\mu m_H} = \beta P \qquad P_{rad} = \frac{1}{3}aT^4 = (1-\beta)P$ for a non-degenerate gas

Equating P_g & P_r & solving for T $\Rightarrow$ $T = \left(\frac{3K}{\mu m_H} \frac{1-\beta}{a\beta}\right)^{1/3} \rho^{1/3}$

Since $P = \frac{K}{\mu m_H} \frac{\rho T}{\beta}$ $\Rightarrow$ $P = \left[\left(\frac{K}{\mu m_H}\right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4}\right]^{1/3} \rho^{4/3}$

In general $\beta = \beta(r)$. However for special configurations for which $\beta =$ const $\Rightarrow$ $P = K\rho^{4/3}$ ie: a polytrope w/ $n=3$. This is close to stars in radiative equilibrium, ie: stars for which the energy is transported by radiative transfer rather than by convection.

3RD EXAMPLE

Stars supported by the pressure of a completely degenerate e^- gas (eg, white dwarfs) $\Rightarrow$ $P \propto$ $\rho^{5/3}$ non-relativistic, $\rho^{4/3}$ relativistic

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Let's define $\rho \equiv \lambda \phi^n$ (since $\rho \propto T^n$ in a non-degenerate polytrope of index n)

λ: scaling parameter whose value depends on the definition of ϕ.

$\Rightarrow P = K\rho^{n+1/n} = K\lambda^{n+1/n}\phi^{n+1}$

From (2) $\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho \Rightarrow M_r = -\frac{r^2}{G}\frac{1}{\rho}\frac{dP}{dr}$

Differentiating this $\frac{dM_r}{dr} = -\frac{1}{G}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) \underset{\text{from (1)}}{\equiv} 4\pi r^2\rho$

$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$

Substituting $P = K\lambda^{n+1/n}\phi^{n+1}$, $\rho = \lambda\phi^n$ $\Rightarrow (n+1)K\lambda^{1/n}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -4\pi G\lambda\phi^n$ ③

Let's define a unit of length $a := \left[\frac{(n+1)K\lambda^{(1-n)/n}}{4\pi G}\right]^{1/2}$

eq $\xi = r/a$ dimensionless distance variable

$$\Rightarrow \boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n}$$

Lane-Emden equation for a polytrope with index n (aka polytropic equation)

The solution $\phi(\xi)$ completely determines the structure of the polytrope except for the choice of P @ $r=0$

By setting $\lambda = \rho(@\, r=0) \Rightarrow \phi$ must obey certain boundary conditions at the center of the star (@ $\xi = 0$) ie: $\left.\begin{array}{l}\phi = 1\\ \frac{d\phi}{d\xi} = 0\end{array}\right\}$ at $\xi = 0$

$$\frac{d\phi}{d\xi} = 0 \leftrightarrow \left.\frac{dP}{dr}\right|_{r=0} = 0$$

The solution ϕ satisfying the Lane-Emden eq. of index n under these boundary conditions is called Lane-Emden function of index n. For $n \neq 0, 1, 5$, numerical techniques must be employed to determine the Lane-Emden function.

<u>NOTE</u>: If $\phi(\xi)$ is a solution, $\phi(-\xi)$ is also a solution.
$\Rightarrow$ if ϕ is expressed as a power series in ξ, only even powers of ξ appear, ie: $\phi(\xi) = C_0 + C_2\xi^2 + C_4\xi^4 + \ldots$

By taking a sufficient number of terms, one can calculate the solution to any accuracy for $\xi < 1$. For $\xi > 1$, the solution can be continued from the differential equation by standard numerical methods.

The solutions decrease monotonically from the center; for $n < 5$, $\phi(\xi = \xi_1) = 0$. When $\xi = \xi_1 \Rightarrow \phi = 0$ & therefore $P(\xi = \xi_1) = 0 \Rightarrow$ the configuration has a physical boundary at $\xi = \xi_1$, ie: final radius of the model.

For $n = 5 \longrightarrow \xi_1 = \infty$

since $\xi = r/a$ & $\lambda = \rho_c$

Each Lane-Emden function ϕ_n represents, for a specific value of K, a one-parameter family of solutions, with the parameter being the central density λ.

NOTE: For example 2, $P = \underbrace{\left[\left(\frac{K}{\mu m_H}\right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4}\right]^{1/3}}_{K} \rho^{4/3}$

w/ β = constant throught the standard model $(\beta = P_g/P)$

$\Rightarrow$ infinite number of solutions determined by the central density $\lambda = \rho_c$.

Radius of polytrope:

$$\boxed{R := \underset{\text{physical boundary}}{a\xi_1} = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \lambda^{(1-n)/2n} \xi_1}$$

Mass of polytrope: w/in radius ξ

$$M(\xi) := \int_0^{a\xi} 4\pi r^2 \rho \, dr = 4\pi a^3 \int_0^{\xi} \lambda \phi^n \xi^2 d\xi$$

using Lane-Emden eq.

$$= -4\pi a^3 \int_0^{\xi} \lambda \frac{d}{d\xi}\left(\xi^2 \frac{d\phi}{d\xi}\right) d\xi$$

$\Rightarrow \quad M(\xi) = -4\pi a^3 \lambda \, \xi^2 \frac{d\phi}{d\xi}$

Substituting a & evaluating at $\xi = \xi_1$, the total mass of the star

$$\boxed{M = -4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \lambda^{(3-n)/2n} \left(\xi^2 \frac{d\phi}{d\xi}\right)_{\xi_1}}$$

NOTE: For $n=3$, M depends only on K & not on λ.

For $n=3$ (standard-model polytrope) $\quad M = 18 \frac{\sqrt{1-\beta}}{\mu^2 \beta^2} M_\odot$

ie: for a given composition μ, the mass determines the value of $\beta = P_g/P$.

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white dwarf: body supported by the pressure of completely degenerate e^-s. As the mass is increased, ρ_c increases so that the degeneracy is relativistic at the center (non relativistic in the outer parts of the star)

$\Rightarrow \quad P_c \rightarrow 1.244 \times 10^{15} \left(\rho/\mu_e\right)^{4/3} \frac{\text{dynes}}{\text{cm}^3}$

As the mass increases & the density rises, the above becomes true everywhere in the WD.

$n=3 \Rightarrow K = 1.244 \times 10^{15} \mu_e^{-4/3}$

$$\Rightarrow \boxed{M_{WD} = \frac{5.83}{\mu_e^2} M_\odot \quad \text{Chandrasekhar limit}}$$

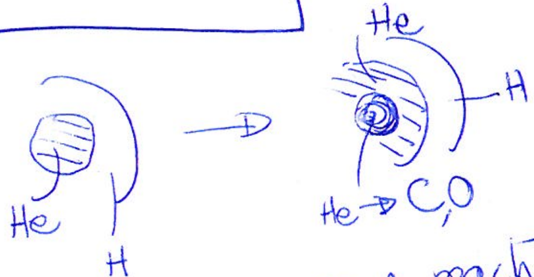
maximum mass that could be supported by e^- degeneracy.

$\mu_e = \frac{2}{1+X}$ e^- mean molecular weight

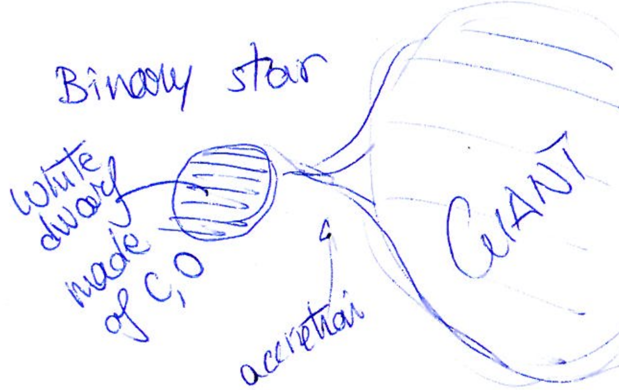
- If gas all H $\longrightarrow$ $X=1$, $\mu_e = 1$ $M = 5.83\,M_\odot$
  This is ~~not~~ physical as it doesn't exist a star all of degenerate & relativistic H.

- If $X \cong 0$ (He core in giant stars; He $\rightarrow$ C,O)
  $\Rightarrow \mu_e = 2 \Rightarrow \boxed{M = 1.4\,M_\odot}$

<u>EXAMPLE</u>: $M_\star = 2\,M_\odot \longrightarrow$



$\Rightarrow$ the core can reach the limit $M \simeq 1.4\,M_\odot$



If the mass transfer is slow, H $\rightarrow$ He $\longrightarrow$ C

$\sim 10^{-8} - 10^{-9}\,\frac{M_\odot}{yr}$

& the mass can grow. As the mass grows, the radius gets smaller, the density increases, & the e^- become relativistic (already degenerate), when $M > 1.4\,M_\odot$, P_e can't support the star any longer $\Rightarrow$ collapse in a dynamical time.

C,O fuse everywhere $\Rightarrow$ SN I, explosion w/ shock wave expanding at supersonic speeds.

<u>NO REMNANT</u> $\Leftarrow$

$$\bar{\rho} = \frac{M}{V} = -\frac{3\lambda}{\xi_1}\left(\frac{d\phi}{d\xi}\right)_{\xi_1} \Rightarrow \frac{\bar{\rho}}{\rho_c} = -\frac{3}{\xi_1}\left(\frac{d\phi}{d\xi}\right)_{\xi_1}$$

$\rho_c = \lambda$

ie: the ratio $\bar{\rho}/\rho_c$ only depends upon the index of the polytrope

indication of the extent to which matter is concentrated toward the center.

$\rho_c = \lambda$

$$P_c = K\lambda^{n+1/n}$$

$$R = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \lambda^{(1-n)/2n} \xi_1 = \left[\frac{(n+1)}{4\pi G}\xi_1^2\right]^{1/2}\left[K\lambda^{(1-n)/n}\right]^{1/2}$$

$$\Rightarrow K\lambda^{(1-n)/n} = \frac{4\pi G R^2}{(n+1)\xi_1^2}$$

$$\Rightarrow P_c = \left(K\lambda^{(1-n)/n}\right)\lambda^2 \underset{\lambda=\rho_c}{=} K\lambda^{(1-n)/n}\rho_c^2 = \quad \rho_c = -\frac{\xi_1}{3}\left(\frac{d\phi}{d\xi}\right)^{-1}_{\xi_1}\bar{\rho}$$

$$= \frac{4\pi R^2 G}{(n+1)\xi_1^2}\left[\frac{\xi_1}{3}\frac{1}{(d\phi/d\xi)_{\xi_1}}\right]^2 \bar{\rho}^2 = \quad \bar{\rho} = M/\tfrac{4}{3}\pi R^3$$

$$\Rightarrow \boxed{P_c = \frac{1}{4\pi(n+1)}\frac{1}{(d\phi/d\xi)^2_{\xi_1}}\frac{GM^2}{R^4}}$$

central pressure expressed in terms of macroscopic quantities.

$n=3$ (standard model)

$$P_c = 1.24\times10^{17}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R_\odot}{R}\right)^4 \frac{\text{dynes}}{\text{cm}^2}$$

more realistic value

The central temperature is computed from P_c & ρ_c by using the appropriate eq. of state.
For ideal ionized non-degenerate gas:

$$P_g = \frac{k \rho_c T_c}{\mu m_H} = \beta_c P_c \quad \text{w/} \quad P_c = \text{gas} + \text{radiation}$$

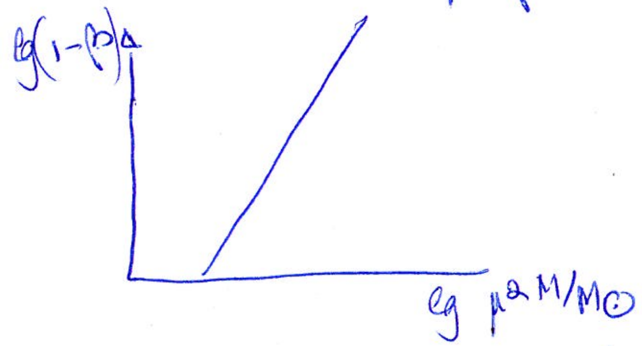
$$\Rightarrow \boxed{T_c = \frac{\mu m_H}{\underset{\text{Boltzmann constant}}{k}} \frac{\beta_c P_c}{\rho_c}}$$

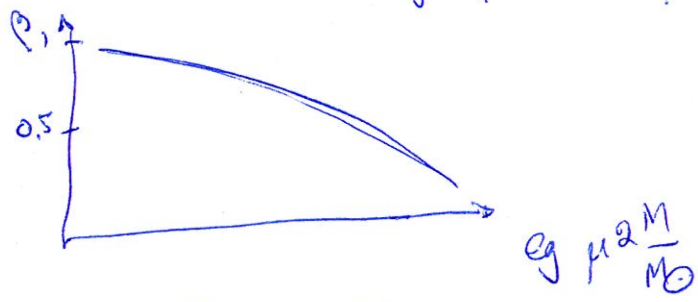
For the standard model ($n=3$) $T_c = 4.6 \times 10^6 \, \mu \beta \left(\frac{M}{M_\odot}\right)^{2/3} \rho_c^{1/3}$

NOTE: For a fixed μ, β is a function of M (or viceversa), i.e.:

$$M = \frac{18 \sqrt{1-\beta}}{\mu^2 \beta^2} M_\odot$$

→ Figures showing dependence of β upon $\mu^2 (M/M_\odot)$





$\Rightarrow$ increasing importance of radiation pressure $(1-\beta)$ with increasing mass.

NOTE: $\mu \sim 0.5 \div 2$

For $n=3$, $\rho_c = 54.2 \, \bar{\rho} \Rightarrow T_c = 17.4 \times 10^6 \mu \beta \left(\frac{M}{M_\odot}\right)^{2/3} \bar{\rho}^{1/3}$

For main-sequence stars, we know L, R, luminosity & radius. From $\rho = M/V$ & the mass-luminosity relation (& $\bar{\rho}_\odot = 1.4$ g/cm^3)

$$\Rightarrow \bar{\rho} \approx \frac{1.4}{M/M_\odot} \text{ g/cm}^3$$ average density in main-sequence stars

The density of main-sequence stars decreases with increasing mass (consequence of the Virial theorem)

From Figure $(1-\beta)$ vs $\mu^2 M/M_\odot$, $\beta \simeq 1$ for main-sequence stars.
For partially depleted H cores $(X \simeq 0.5, Y \simeq 0.5)$, $\mu = 0.7$

$$\Rightarrow \boxed{T_c \simeq 14 \times 10^6 \left(\frac{M}{M_\odot}\right)^{1/3}}$$ central temperature for main-sequence stars.

NOTE: There are only three analytic solutions to the Lane-Emden eq.

$$n = 0 \longrightarrow \phi(\xi) = 1 - \frac{\xi^2}{6} \quad w/ \ \xi_1 = \sqrt{6}$$

$$n = 1 \longrightarrow \phi(\xi) = \frac{\sin \xi}{\xi}, \quad w/ \ \xi_1 = \pi$$

$$n = 5 \longrightarrow \phi(\xi) = \left[1 + \xi^2/3\right]^{-1/2} \quad w/ \ \xi_1 \to \infty$$

NOTE: For $n = 3$

$$R = 5.45749 \left(\frac{M}{\rho_c}\right)^{1/3} \Rightarrow \frac{R}{R_\odot} = 9.86 \left(\frac{M}{M_\odot}\right)^{1/3} \frac{1}{\rho_c^{1/3}}$$