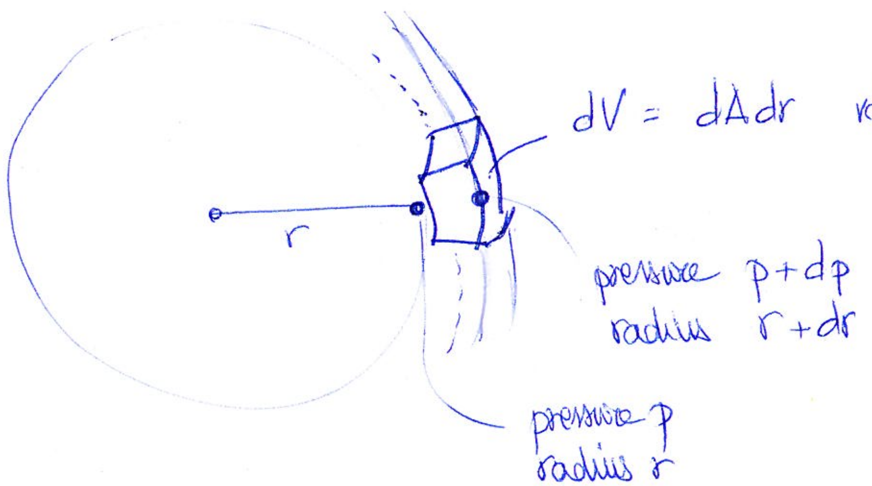
STELLAR STRUCTURE

Consider the problem of an isolated, non-rotating, non-magnetic spherical mass of gas held together by gravity.
(BUT stars do rotate, frequently occur in binary pairs, and have observable magnetic fields $\longrightarrow$ as perturbations of realistic model)

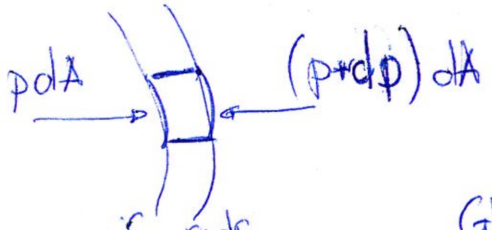
- sphere of gas in hydrostatic equilibrium (meccanic equilibrium)



$$M_r = \int_0^{r'} 4\pi r^2 \underset{\text{density}}{\rho(r)}\, dr \quad \overset{\text{equivalent to}}{\Longleftrightarrow} \quad \boxed{\underset{\text{mass gradient}}{\frac{dM_r}{dr}} = 4\pi r^2 \rho(r)} \quad (1)$$

Equation of conservation of mass

$$\underbrace{\rho \ddot{r}}_{\sim m \cdot a} = \underbrace{F_g + F_{p_r} + F_{p_{r+dr}}}_{F}$$



$\sim m \cdot a$
mass · acceleration

$\ddot{r} := \frac{d^2 r}{dt^2}$ (acceleration)

w/ $F_g < 0$ gravitational force inward $F_g = -\frac{GM_r}{r^2}\rho$

$F_{p_r}, F_{p_{r+dr}}$ pressure forces from the left & right (above & below).

$\Longrightarrow$ Pressure force effecting on the volume element $F_p = \frac{p\,dA - (p+dp)\,dA}{dV} = -\frac{dp}{dr}$

$$\boxed{\rho\ddot{r} = \underbrace{-\frac{GM_r}{r^2}\rho}_{\text{acceleration due to gravity}} - \frac{dP}{dr}}$$ Equation of motion

pressure gradient (balancing the gravitational pull)

IF hydrostatic equilibrium $\Rightarrow \rho\ddot{r} = 0$

$$\boxed{\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho} \quad (2)$$ EQUATION OF HYDROSTATIC EQUILIBRIUM (conservation of momentum)

NOTE: The properties of central temperature & pressure, i.e.: T_c & P_c, are determined by the eq. of hydrostatic equilibrium (2)
If mean quantities are considered

$dP \simeq \Delta P = P_c - P_s \approx P_c$ since $P_c \gg P_s$

P_c central pressure, P_s surface pressure

$d \simeq R$

$\Rightarrow \frac{dP}{dr} \simeq \frac{P_c}{R} = \frac{GM}{R^2}\frac{M}{\frac{4}{3}\pi R^3} \Rightarrow P_c = \frac{3}{4\pi}\frac{GM^2}{R^4}$

$\langle\rho\rangle = M/V$

$G = 6.67384 \times 10^{-8} \frac{cm^3}{g\, s^2}$

$M_\odot = 1.9891 \times 10^{33}$ g

$R_\odot = 6.955 \times 10^{10}$ cm

$\rightarrow P_c|_\odot \simeq 3 \times 10^{15} \frac{g}{cm\, s^2} = 3 \times 10^{15}$ Ba (large)

$\doteq 3 \times 10^{15} \frac{dine}{cm^2}$

$1\,atm = 1 \times 10^6 \frac{dyne}{cm^2}$ 1 dyne $= 1 \frac{g\,cm}{s^2}$
(units of surface tension)
1 dyne $= 10^{-5}$ N

$\simeq 3 \times 10^9$ atmospheres

$$P_c = -\int_{R_s}^{R_c} \frac{GM_r}{r^2}\rho\, dr$$

$P_c \sim 2 \times 10^{11}$ atm!! ENORMOUS CENTRAL PRESSURE

FROM (2) $\Rightarrow$ in order for a star to be static, a <u>pressure gradient</u> dP/dr must exist from the center to the surface to counteract the force of gravity - it is the pressure gradient that supports the star (not the pressure itself)

Since $M_r \neq 0$ except in the center $\rightarrow \frac{dP}{dr}$ is always finite or null, & it is continuous.

$$\left.\frac{dP}{dr}\right|_c = 0 \quad \left(\text{since } \left.M_r\right|_c = 0\right)$$

~~PRESSURE~~ EQUATION OF STATE

$$\boxed{\text{IDEAL GAS LAW}: \quad P = nKT}$$

n ↑ # density of particles

T ↑ temperature

$n = N/V$

For hydrogen $\quad n = \frac{\langle\rho\rangle}{m_p} \quad$ w/ $\langle\rho\rangle = \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3} \simeq 1 \frac{g}{cm^3}$

(like the density of the water!)

$$\rightarrow T_c \simeq \frac{P_c}{nK} = \frac{P_c\, m_p}{\langle\rho\rangle K} \simeq 2 \times 10^7 \, {}^\circ K$$

$$\simeq 20 \text{ million } K$$

<u>NOTE:</u> The eq. of hydrostatic eq. (2) w/ the equation of state of a perfect gas give us T_c & P_c, huge information!!

T_c is very huge $\Rightarrow$ the assumption of a perfect gas is correct, since H & He are completely ionized (above $10^5 K$) & many other elements are ionized above $10^6 K$.

PLASMA, ions & e^-s $\Leftarrow$

Lower Limit on P_c: $\frac{dP}{dr} = -\rho \frac{GM_r}{r^2}$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho(r) \quad \Bigg\} \Rightarrow \frac{dP}{dr} = -\frac{GM_r}{4\pi r^4}\frac{dM}{dr}$$

Since $\frac{d}{dr}\left(P + \frac{GM_r^2}{8\pi r^4}\right) = \underbrace{\frac{dP}{dr} + \frac{GM_r}{4\pi r^4}\frac{dM}{dr}}_{=0 \text{ from eq. of hydrostatic equilibrium.}} \underbrace{- \frac{GM_r^2}{2\pi r^5}}_{<0}$

$$\Rightarrow \frac{d}{dr}\left(P + \frac{GM_r^2}{8\pi r^4}\right) < 0$$

$$\Rightarrow P_c > \frac{GM^2}{8\pi R^4} = 4.4\times 10^{14}\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)^4 \frac{\text{dyne}}{\text{cm}^2}$$

$$P_{c,\odot} \gtrsim 5\times 10^{14}\ \frac{\text{dyne}}{\text{cm}^2}.$$

IF $\rho\ddot{r} \neq 0 \Rightarrow$ no equilibrium between force of gravity & pressure

$$\Rightarrow \underbrace{\frac{d^2r}{dt^2}}_{\text{acceleration}} \simeq \underbrace{-\frac{GM_r}{r^2}}_{\text{acceleration of gravity}} = -\frac{4\pi G\rho}{3} r \text{ (assuming no pressure)}$$

eq. of harmonic oscillator w/ frequency $f = \frac{1}{T} = \sqrt{K}/2\pi$

Dynamical time $t_{dyn} = T/4$

$\Rightarrow$ Dynamical time $t_{dyn} = \sqrt{\frac{3\pi}{16 G\rho}} = \sqrt{\frac{\pi^2 R^3}{4GM}} \simeq 2500$ sec for the Sun $\simeq \frac{3}{4}$ hr FAST!

This is independent on the amount of non-equilibrium. Hence, in the Sun, hydrostatic equilibrium is maintained w/ high precision.

for cepheids $\sim$ a few days

VIRIAL THEOREM (an alternative approach employing macroscopic quantities)

Hydrostatic equilibrium $\longrightarrow \frac{dP}{dr} = -\rho \frac{GM(r)}{r^2}$

Multiplying by $V(r)dr = \frac{4}{3}\pi r^2 dr$

$$\rightarrow V(r)\, dP = -\frac{1}{3} 4\pi r^2 \rho \, dr \frac{GM(r)}{r} = -\frac{1}{3}\frac{GM}{r} dM$$

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

$$\rightarrow \int_0^R V(r)\,dP = -\frac{1}{3}\int_0^M \frac{GM_r}{r} dM$$

$$\underbrace{\int_0^R V(r)\,dP}_{\int_0^R d(P\cdot V) - \int_0^R P\,dV}$$

$$\underbrace{\int_0^R d(P\cdot V)}_{PV\Big|_0^R = 0}$$

since $P\Big|_{=} = 0$, $V\Big|_{r=0} = 0$

$$\Rightarrow -\int_0^R 3P\,dV = \underbrace{-\int_0^M \frac{GM_r}{r} dM}_{\Omega \text{ gravitational potential energy (definition)}}$$

$$\Rightarrow \boxed{-3\int P\,dV = \Omega}$$

For non-relativistic particles $P = \frac{2}{3} K$ ← internal kinetic energy density, ie: $\mathcal{K} = K\cdot V$

For relativistic particles $P = \frac{1}{3}K$

$$\Rightarrow -3\int \frac{2}{3} K\,dV = 2\mathcal{K} \text{ (non relativistic)} \qquad -3\int \frac{1}{3} K\,dV = \mathcal{K} \text{ (relativistic)}$$

L kinetic energy

VIRIAL THEOREM $\boxed{2K + \Omega = 0}$ NON-RELATIVISTIC PARTICLES

($K + \Omega = 0$ for relativistic particles)

ie: for the Sun, $K = -\Omega/2$, ie: only half of the gravitational potential energy goes into kinetic energy (heating)

The total energy of a monoatomic gas

$E = \Omega + K = \Omega/2$ (non-relativistic)

ie: the Sun is in a bound state.

$E = 0$ relativistic, ie: it is unstable.

NOTE: If radiation pressure dominates ($P = \frac{1}{3} a K T^4$) over $P = nKT \Rightarrow E = 0$.

NOTE: If the star is contracting slowly, so that I can apply the virial theorem (ie: hydrostatic equilibrium is valid) $\Rightarrow 2\,dK + d\Omega = 0$

$\Rightarrow$ half of the gravitational potential energy goes into internal kinetic energy (heating up the interior), while the other half is emitted as radiation from the star (luminosity).

———— o ————

A star cannot be in perfect hydrostatic equilibrium, as a net outward flow of energy occurs through the star, & T varies w/ location. As gas particles collide w/ one another & interact w/ the radiation field by absorbing & emitting photons, the description of the processes of excitation & ionization became complex.

IF the distance over which the temperature changes >> distances traveled by the particles & photons between collisions (ie: their MEAN FREE PATHS)

$\Rightarrow$ $\boxed{\text{LOCAL THERMODYNAMIC EQUILIBRIUM (LTE)}}$ ★

Temperature scale height $\quad H_T := \dfrac{T}{|dT/dr|}$

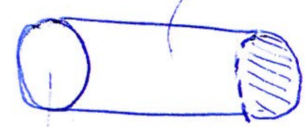
characteristic distance over which the Temperature varies

Average thermal gradient $:= \dfrac{T_c}{R} = \dfrac{2\times10^7\,K}{7\times10^{10}\,cm} \simeq 10^{-4}\,\dfrac{K}{cm}$

$$H_T\Big|_{\odot}^{\text{photosphere}} = \frac{5\,800\ K}{10^{-4}\ K/cm} \approx 600\ Km \text{ in the photosphere}$$

MEAN FREE PATH

V = σ v



σ transversal cross-section (area)

$$\lambda := \underbrace{\tau_{collision}}_{\text{time between collisions/interactions}} \cdot \; v$$

v: velocity of particles

$\sim \dfrac{1}{N}$ w/ N # of collisions/interactions in the unit of time $[N] = \#/s$

$N = n \cdot \sigma v$ — σv: volume travelled in the unit of time

n: # density w/in the volume σv spanned in the unit of time

$\Rightarrow$ $\boxed{\lambda = \frac{1}{n\sigma}}$ mean free path (m.f.p.) w/ n particle density, σ interaction cross-section. (8)

<u>Ex 1</u>: For a plasma (medium completely ionized) w/ numbers of e^-s equal to the number of ions, $n = n_e = \rho / m_p$ (mass dominated by the mass of protons)

For Thompson scattering (elastic scattering of photons by charged particles in this case e^-)

i.e.: low-energy Compton scattering

$$\Rightarrow \sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 0.665 \times 10^{-24}\ cm^2$$

$$n_e = \frac{1\ g/cm^3}{1.67 \times 10^{-24}\ g} \simeq 6 \times 10^{23}\ cm^{-3} \quad \approx 2.5\ cm$$

$$\Rightarrow \lambda = \frac{1}{n_e \sigma_T} \approx 2.5\ cm$$ average mfp inside the Sun

<u>Ex 2</u>: In the photosphere, $\rho = 2.1 \times 10^{5}\ \frac{g}{cm^3}$, primarily of HI atoms in the ground state.

$$n_{HI} = \rho / m_p = 1.25 \times 10^{17}\ cm^{-3}$$

$$\sigma \simeq \pi (2a_0)^2 \simeq 3.52 \times 10^{-16}\ cm^2$$

a_0 Bohr radius (classical approximation)

$$\Rightarrow \ell_{HI} = \frac{1}{n_{HI}\sigma} = 0.02\ cm$$

$2a_0$

⇒ Inside the Sun, $\lambda \sim 2\text{-}3$ cm $\ll H_T$

$\lambda_{phot.} \sim 0.02$ cm

→ LTE

$$\langle dT \rangle \simeq \langle \frac{dT}{dr} \rangle \cdot \lambda = 10^{-4} \frac{°K}{cm} \cdot 2.5 \text{ cm} \sim 2 \times 10^{-4} °K$$

i.e.: the levels of the Sun in contact through the photons (Compton scattering) have $\Delta T \simeq 10^{-4}$ °K, i.e.: very similar temperature. In the Sun, the temperature along the mfp (of the agents responsible for the exchange of energy) does not change much

⇒ L.T.E.

In the sun, there is no global thermal equilibrium, but there is LTE.

(*)

LTE Hydrostatic equilibrium Fully ionized material	⇒ chemical equilibrium

In the stars, we can assume LOCAL THERMODYNAMIC EQUILIBRIUM, i.e.: we can consider small enough volumes so that LTE is valid, a large enough volumes to have enough particles to consider macroscopic properties.

— o —

(1) $\frac{dM_r}{dr} = 4\pi r^2 \rho(r)$ EQ. of CONSERVATION OF MASS

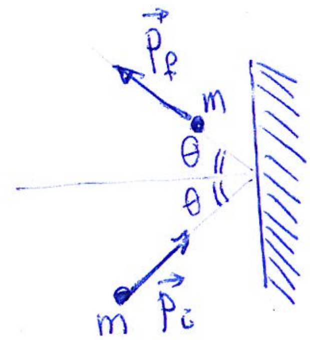
(2) $\frac{dP}{dr} = -\frac{G M_r \rho}{r^2}$ EQ. of HYDROSTATIC EQUILIBRIUM

I need an equation of state to solve this set of equations.

The pressure integral

The microscopic source of pressure in a perfect gas is particle bombardment. $\Rightarrow$ transfer of momentum, hence a force ($F = dp/dt$). The average force per unit area is the pressure.

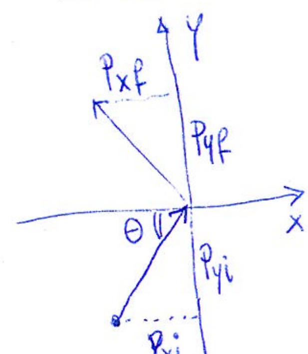
In thermal equilibrium in stellar interiors (LTE), the angular distribution of particle momenta is isotropic, ie: particles are moving w/ equal probability in all direction.



Variation of the momentum of the particle m:

$\Delta p_x = p_{xf} - p_{xi} = -2p_{xi}$ $\Delta p_y = 0$

since the final component $p_{xf} = -p_{xi}$ (ie: it changes sign)



$P_{pressure} := \frac{\text{Force}}{\text{Area}}$ Force $:= \frac{\Delta p}{\Delta t}$ ← momentum

Momentum transferred to the surface is $-\Delta p_x$

We can also write: $\Delta p = 2p\cos\theta$ momentum transferred to the surface.

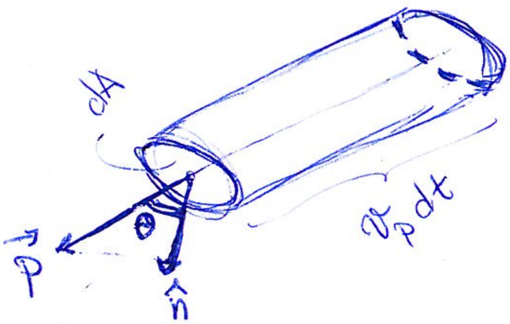
Let $F(\theta, p)\, d\theta\, dp :=$ # particles w/ momentum p in the range dp striking the surface per unit of area per unit of time from all directions inclined at angle θ to the normal in the range $d\theta$

$$dP := \frac{\Delta p}{\Delta t \, \Delta \text{area}} = 2p\cos\theta \; F(\theta, p)\, d\theta\, dp$$

contribution to pressure from those particles

$$F(\theta,p)\,d\theta\,dp = \underbrace{v_p \cos\theta}_{\substack{\text{volume of such particles capable of passing}\\ \text{through the unit surface in unit of time}}} \; \underbrace{n(\theta,p)\,d\theta\,dp}_{\text{\# density of particles moving in the prescribed cone}}$$

$n(\theta,p)$: # density of particles w/ momentum p in the range dp moving in the cone of directions inclined at angle θ in the range $d\theta$

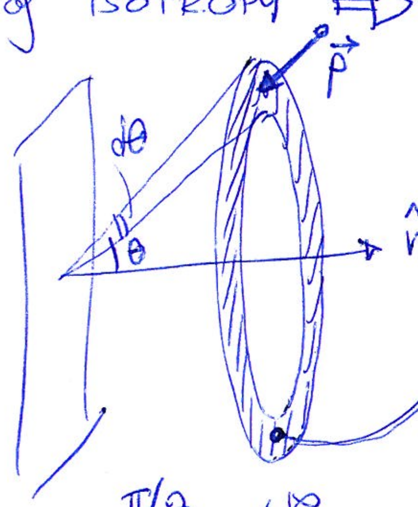


$$\hookrightarrow V = v_p dt \cdot dA \cdot \cos\theta$$

$$\Rightarrow \frac{V}{dA\,dt} = v_p \cos\theta$$

Because of ISOTROPY $\Rightarrow$

$$n(\theta,p)\,d\theta\,dp = \frac{2\pi \sin\theta\, d\theta}{4\pi} \underbrace{n(p)\,dp}_{\text{Total number density of particles of momentum } p \text{ in } dp}$$



$$\underbrace{2\pi \sin\theta\, d\theta}_{d\Omega}$$

solid angle

NOTE: $d\Omega = \sin\theta\, d\theta\, d\phi$, w/ $\phi \in [0, 2\pi]$, $\theta \in [0, \pi/2]$ semi-sphere

$$\Rightarrow P = \int_{\theta=0}^{\pi/2} \int_0^\infty 2p\cos\theta \; v_p \cos\theta \; \frac{2\pi\sin\theta}{4\pi}\, d\theta \; n(p)\, dp$$

$$= \int_0^{\pi/2}\int_0^\infty p\, v_p\, n(p) \cos^2\theta \sin\theta\, d\theta\, dp = \int_0^\infty p\, v_p\, n(p)\, dp \overbrace{\int_0^{\pi/2} \cos^2\theta\, d(\cos\theta)}^{1/3}$$

$$\Rightarrow P = \frac{1}{3}\int_0^\infty p\, v_p\, n(p)\, dp$$ PERFECT, ISOTROPIC GAS (pressure integral)

NOTE: the relation between p & v_p depends upon relativistic considerations, whereas $n(p)$ depends on the type of particles & quantum statistics.

1) PERFECT, MONOATOMIC, NON ~~DEGENERATE~~ GAS.

For non-relativistic particles $\rightarrow p = mv$; $n(p)\,dp = n(v)\,dv$

$$\Rightarrow P = \frac{1}{3}\int_0^\infty m\, v^2\, n(v)\, dv = \frac{1}{3} m \int_0^\infty v^2\, n(v)\, dv$$

$$n(v)\,dv = n\left(\frac{m}{2\pi KT}\right)^{3/2} e^{-mv^2/2KT}\, 4\pi v^2\, dv$$

Maxwell-Boltzmann distribution

$$\int_0^\infty v^2\, n(v)\, dv =: n\cdot \overline{v^2} = \frac{3KT}{m}\, n$$

for a Maxwell-Boltzmann distribution

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{\frac{3KT}{m}}$$

$$\Rightarrow P = \frac{1}{3}\cancel{m}\cdot n\,\frac{3KT}{\cancel{m}} \Rightarrow \boxed{P = nKT}$$

w/ n # density of particles

$n = \rho/\overline{m}$, w/ $\overline{m}$ average mass of gas particles

$$\Rightarrow P = \frac{\rho KT}{\overline{m}}$$

NOTE: Relativistic case

$$v = \frac{p/m}{\sqrt{1+(p/mc)^2}}$$

Mean molecular weight of the perfect gas $\mu := \frac{\bar{m}}{m_H}$

w/ $m_H = 1.67353249 \times 10^{-24}$ g

$$\Rightarrow \boxed{P = \frac{\rho K T}{\mu m_H}} = \frac{N_0}{\mu} \rho K T$$

Avogadro's number

$N_0 := \frac{1}{m_H} = 6.0225 \times 10^{23}$ mole^{-1}

μ depends on the composition of the gas & on the ionization state of each species (free e^-'s must be included in the calculation of $\bar{m}$)

↳ need to use Saha's equation

$$\frac{1}{\mu} = \frac{X \cdot n_H}{1.008} + \frac{Y \cdot n_{He}}{4.004} + \underbrace{(1 - X - Y)}_{Z} \left\langle \frac{n_Z}{A_Z} \right\rangle$$

average of n_Z / A_Z for $Z > 2$, w/ n_Z # of free particles contributed to the gas by the atom of element Z w/ atomic number $A_j = \frac{m_j}{m_H}$

For solar abundances, $\left\langle \frac{n_Z}{A_Z} \right\rangle \approx \frac{1}{15.5}$ for a neutral gas ($n_Z = 1$)

X: = weight fraction of H

Y: = weight fraction of Y

Z: = weight fraction of all species heavier than He

NOTE: for complete ionized gas,

$n_Z = Z + 1$ (e^-'s, & nucleus)

for complete neutral gas, $n_Z = 1$

FOR COMPLETELY NEUTRAL GAS: $\left.\frac{1}{\mu}\right|_{n} \approx X + \frac{Y}{4} + Z \left\langle \frac{1}{A_Z} \right\rangle$

In the interior of a star, when the gas is completely ionized,

$n_H = 2$, $n_{He} = 3$, $n_{Z_i} = 1 + Z_i$

The average atomic weight of element Z is $A_Z \approx 2Z + 2$
(when fraction by weight of the species heavier than He is small)

$$\Rightarrow \left.\frac{1}{\mu}\right|_i \simeq 2X + \frac{3}{4}Y + \frac{Z}{2} \underset{Z = 1-X-Y}{=} \frac{1}{2}(3X + 0.5Y + 1)$$

COMPLETELY IONIZED GAS

For young stars $X = 0.7$, $Y = 0.28$, $Z = 0.02$ $\Rightarrow$ $\mu_n \sim 1.30$, $\mu_i \sim 0.62$

__For electrons:__

$$n_e = \frac{\rho}{m_H}\left[X(n_H - 1) + \frac{Y}{4}(n_{He} - 1) + (1-X-Y)\left\langle \frac{n_Z - 1}{A_Z} \right\rangle\right]$$

density of e^-s

For complete ionization, $n_Z = 1 + Z_i$ $\longrightarrow$ $n_H - 1 = 1$, $n_{He} - 1 = 2$, $n_Z - 1 = n_{Z_i}$

$$\Rightarrow \left.n_e\right|_i = \frac{\rho}{m_H}\left[X + \frac{Y}{2} + \underbrace{(1-X-Y)}_{Z_i}\underbrace{\left\langle \frac{n_Z}{A_Z} \right\rangle}_{\approx \frac{1}{2}}\right]$$

when Z_i is small

$$\Rightarrow \left.n_e\right|_i \simeq \frac{1}{2}\frac{\rho}{m_H}(X + 1)$$

(IMPORTANT for the Chandrasekhar mass limit)

Mean molecular weight per electron $\mu_e := \left[X(n_H - 1) + \frac{Y}{4}(n_{He} - 1) + (1-X-Y) \cdot \left\langle \frac{n_Z - 1}{A_Z} \right\rangle\right]$

$$\Rightarrow n_e = \frac{\rho}{\mu_e m_H}, \quad \left.\mu_e\right|_i = \frac{2}{X+1}$$

It is common to think of the chemical composition of the solar system as a standard, against which other compositions are to be compared.

$$\left[\frac{Fe}{H}\right] := \lg \frac{n_{Fe}/n_H|_{star}}{n_{Fe}/n_H|_{\odot}}$$

(since Iron is observable spectroscopically)

For Pop I stars, $X \simeq 0.70$
$Y \simeq 0.28$
$Z \simeq 0.02$

NOTE: Since on the main-sequence, $L \propto \mu^{7-7.5}$, a small change in μ corresponds to large changes in the luminosities.

There are 2 extremely important physical cases for which the eq. of state of a perfect, non degenerate monoatomic gas is not adequate:

A) the pressure due to photons in the star interior is comparable to the pressure due to particles

B) the electron gas becomes degenerate.

NOTE: From the Maxwell-Boltzmann equation

$$\overline{v^2} = \frac{3KT}{m} \Rightarrow \underbrace{\frac{1}{2} m \overline{v^2}}_{\substack{\text{average Kinetic}\\ \text{energy per}\\ \text{particle}}} = \frac{3}{2} KT$$

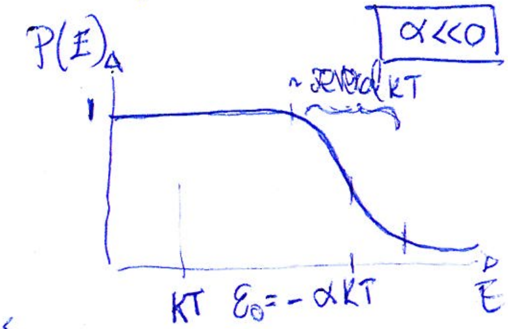
(w/ 3 degrees of freedom, ie: isotropy)

The average kinetic energy per particle per degree of freedom is $\frac{1}{2}KT$.

2) ELECRON DEGENERACY.

e^-_s obey the Fermi-Dirac statistics (particles w/ half-integer spin)

$$\Rightarrow \underbrace{n_e(p)\,dp}_{\substack{\text{density of } e^-_s \text{ w/} \\ \text{momentum } p \text{ in } dp}} = \underbrace{\frac{2}{h^3} 4\pi p^2\,dp}_{g(p)\,dp} \cdot \underbrace{\frac{1}{e^{\alpha + \frac{E}{kT}} + 1}}_{\substack{P(p) \\ \text{occupation index} \\ \text{for a Fermi gas}}}$$



NOTE: $\boxed{\alpha \gg 0}$ (so that $N = \int_0^\infty n_e(p)\,dp$ is small) — total # density

$\max P(p) = 1$ (due to Pauli exclusion principle)

$\Rightarrow$ Fermi statistics $\simeq$ Maxwell-Boltzmann statistics $\Rightarrow$ non-degenerate, ideal gas case.

$$n_e(p)\Big|_{MAX} = \frac{2}{h^3} 4\pi p^2$$

maximum density of e^-_s in momentum space

$\Rightarrow$ this restriction on $n_e(p)$ creates degeneracy pressure.

If one increases n_e, the e^-_s are forced into high-lying momentum states because the lower states are occupied, making a large contribution to the pressure integral.

For any given temperature T & n_e $\Rightarrow$ $n_e = \int_0^\infty n_e(p)\,dp = n_e(\alpha, T)$.

$$\Downarrow$$

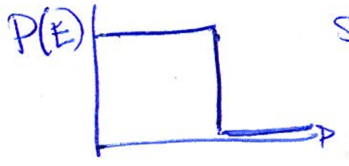
$$\boxed{\alpha}$$

If n_e is increased @ constant T, α becomes small, becoming $\alpha \ll 0$ for high density.

For $\alpha \ll 0$ $\Rightarrow$ $P(p) = \begin{cases} 1 & \text{for } E/kT < |\alpha| \\ 0 & \text{for } E/kT > |\alpha| \end{cases}$

This transition occurs smoothly over several kT near $E = |\alpha| kT$.

For $-\alpha KT >> KT$ $\Rightarrow$



step function $\Rightarrow$ COMPLETE DEGERACY

& $E_f = |\alpha| KT$ is the Fermi energy.

2A) COMPLETE DEGENERACY.

$\rightarrow$ The density is high enough that all available e^- states w/ energies less than some maximum energy are filled.

Since n_e = finite $\Rightarrow$ $n_e(p)\,dp = \begin{cases} \frac{2}{h^3} 4\pi p^2\,dp & \text{for } p<p_0 \\ 0 & \text{for } p>p_0 \end{cases}$

NOTE: complete degeneracy can be seen as the state of minimum kinetic energy, the ground state of a degenerate perfect e^- gas.

$$\Rightarrow \quad n_e = \int_0^\infty n_e(p)\,dp = \int_0^{p_0} n_e(p)\,dp = \frac{8\pi}{3h^3} p_0^3$$

$\Rightarrow$ Maximum momentum of a completely degenerate gas can be calculated from n_e, ie: $p_0 = \left(\frac{3h^3}{8\pi} n_e\right)^{1/3}$, to which the Fermi energy is associated.

$$P_e = \frac{1}{3}\int_0^\infty p\, v_p\, n_e(p)\,dp$$

pressure

non-relativistic / relativistic degenerate e^- gas.

NON-RELATIVISTIC COMPLETE DEGENERACY

if $E_0 << m_e c^2 = 0.51$ MeV $\longrightarrow$ $v_p = p/m$

$$\Rightarrow P_e = \frac{8\pi}{3h^3 m} \underbrace{\int_0^{p_0} p^4 dp}_{\frac{1}{5}p_0^5} \Rightarrow P_{e,nr} = \frac{8\pi}{15 m h^3} p_0^5$$

Using $p_0 = \left(\frac{3h^3}{8\pi} n_e\right)^{1/3}$ $\Rightarrow$ $\boxed{P_{e,nr} = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3}}$

NOTE: Eq. of state independent of T $\leftarrow$

$$P_{e,nr} = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{\rho}{\mu_e}\right)^{5/3}\left(\frac{1}{m_H}\right)^{5/3}$$

$$= 1.004 \times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3} \frac{\text{dynes}}{\text{cm}^2}$$

$\mu_e \approx 2$ (unless the gas has considerable amounts of H)

NOTE: $P_{e,nr} \propto n_e^{5/3}$

$P_{\substack{\text{non-degenerate} \\ e^-\text{s gas}}} \propto n_e$

$\Rightarrow$ as n_e increases, $P_{e,nr} > P_{\text{non degenerate}}$

By equating $P_{\text{non-deg}} = P_{e,nr}$ $\longrightarrow$ boundary line in the ρT plate dividing it into regions of degenerate & non-degenerate gas

$$\Rightarrow \frac{k\rho T}{\mu_e m_H} = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{\rho}{\mu_e}\right)^{5/3}\left(\frac{1}{m_H}\right)^{5/3}$$

$\Rightarrow$ $\boxed{P_{e,nr} > P_{e,\text{non deg.}} \text{ when } \frac{\rho}{\mu_e} > 2.4 \times 10^{-8}\, T^{3/2}\ \text{g/cm}^3}$

solving numerically.

NOTE: The transition from non-degenerate to degenerate occurs smoothly. In this transition region ⇒ partial degeneracy

NOTE: Center of Sun $\rho/\mu_e \approx 10^2$, $T \approx 10^7$ ⇒ non-degenerate electron pressure

white dwarf $\rho/\mu_e \approx 10^6$, $T \simeq 10^6$ ⇒ complete degeneracy pressure domination

RELATIVISTIC COMPLETE DEGENERACY

As n_e increases, p_0 grows larger to the point that e^-s in the degenerate distribution become relativistic.

Total energy particle $W^2 = p^2c^2 + m_0^2c^4$, w/ m_0c^2 rest-mass energy + kinetic energy

equivalently $W = mc^2 = \dfrac{m_0 c^2}{1-\left(\frac{v}{c}\right)^2}$ ⇒ $pc = \frac{v}{c} W$

If $p_0 c \sim 2 \underbrace{m_0 c^2}_{\text{rest-mass energy}} \approx 1\,\text{MeV}$ ⇒ relativistic

$$p_0 c = hc\left(\frac{3}{8\pi} n_e\right)^{1/3} = 6.12 \times 10^{-11}\, n_e^{1/3}\ \text{MeV}$$
$$= 5.15 \times 10^{-3} \left(\frac{\rho}{\mu_e}\right)^{1/3}\ \text{MeV}$$

⇒ for $\rho/\mu_e > 7.3 \times 10^6\ \text{g/cm}^3$ relativistic (as in white dwarfs)

$$P_e = \frac{1}{3}\int_0^\infty p\, v_p\, n_e(p)\, dp \quad \text{w/} \quad p = \frac{m_0 v}{\left[1-(v/c)^2\right]^{1/2}} \rightarrow v = \frac{p/m_0}{\sqrt{1+(p/m_0c)^2}}$$

$$P_{e,r} = \frac{1}{3}\int_0^{\infty} \frac{p^2}{m_0} \frac{1}{\sqrt{1+(p/m_0c)^2}}\, n_e(p)\, dp = \frac{1}{3}\int_0^{p_0} \frac{p^2}{m_0\left[1+(p/m_0c)^2\right]^{1/2}} \frac{2}{h^3} 4\pi p^2 dp$$

$$= \frac{8\pi}{3h^3 m_0} \int_0^{p_0} \frac{p^4\, dp}{\left[1+(p/m_0c)^2\right]^{1/2}}$$

if e^-'s highly relativistics $\Rightarrow$ $p/m_0c \gg 1$

$$\Rightarrow P_{e,r} = \frac{8\pi}{3h^3 m_0}\int_0^{p_0} \frac{p^4\, dp}{p/m_0c} = \frac{2}{3}\frac{\pi c}{h^3} p_0^4$$

$$p_0 = \left(\frac{3h^3}{8\pi} n_e\right)^{1/3}$$

$$\Rightarrow \boxed{P_{e,r} = \frac{ch}{8}\left(\frac{3}{\pi}\right)^{1/3} n_e^{4/3}}$$ for a highly relativistic, completely degenerate e^- gas

ie: $P_{e,r} \propto n_e^{4/3}$

Correct solution: $\sinh\theta = p/mc$

$dp = mc\cosh\theta\, d\theta$

$\sinh x := \frac{e^x - e^{-x}}{2} = \frac{e^{2x}-1}{2e^x}$

$\cosh x := \frac{e^x + e^{-x}}{2} = \frac{e^{2x}+1}{2e^x}$

$$\Rightarrow P_{e,r} = \frac{8\pi m_0^4 c^5}{3h^3}\int_0^{\theta_0} \frac{\sinh^4\theta\, d\theta\, \cancel{\cosh\theta}}{\cancel{\left[1+\sinh^2\theta\right]^{1/2}}}$$

$$\left[1+\sinh^2\theta\right]^{1/2} = \cosh\theta$$

$$P_{e,r} = \frac{8\pi m_0^4 c^5}{3h^3}\int_0^{\theta_0} \sinh^4\theta\, d\theta$$

$$\Rightarrow \boxed{P_{e,r} = \frac{8\pi m_0^4 c^5}{3h^3}\left(\frac{\sinh^3\theta_0 \cosh\theta_0}{4} - \frac{3\sinh 2\theta_0}{16} + \frac{3}{8}\theta_0\right)}$$

Written in terms of the Fermi momentum $p_0 = \left(\frac{3h^3}{8\pi} n_e\right)^{1/3}$

$$\Rightarrow \quad P_{e,r} = \frac{\pi m_0^4 c^5}{3h^3} f(x) = 6.003 \times 10^{22} f(x) \ \frac{\text{dynes}}{\text{cm}^2}$$

w/

$$x = p_0/m_0 c = \frac{h}{m_0 c}\left(\frac{3}{8\pi} n_e\right)^{1/3} \equiv 1.195 \times 10^{-10} n_e^{1/3} = 1.009 \times 10^{-2} \left(\frac{\rho}{\mu_e}\right)^{1/3}$$

$$f(x) = x(2x^2 - 3)(x^2+1)^{1/2} + 3 \sinh^{-1} x$$

<u>NOTE:</u> For $x \to 0$, ie: $p_0 \ll m_0 c$ (non-relativistic case)

$$\Rightarrow f(x) \approx \frac{8}{5} x^5 - \frac{4}{7} x^7 + \ldots$$

For $x \to \infty$, $f(x) \approx 2x^4 - 2x^2 + \ldots$

(see table 2-2)

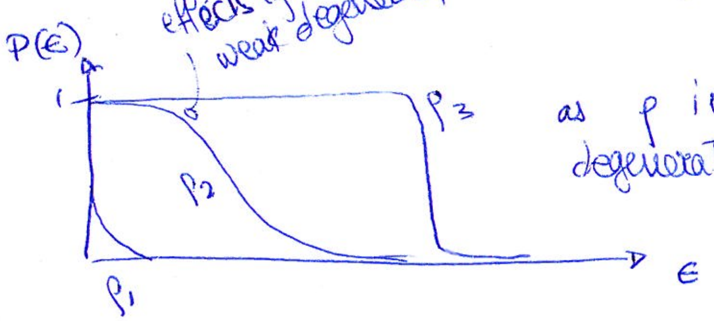
<u>NOTE:</u> To be relativistic, $\rho > 10^6 \ g/cm^3$ for a degenerate gas

At these densities, degeneracy is essentially complete unless $T > 10^9$ K (from $\rho/\mu_e > 2.4 \times 10^{-8} \, T^{3/2} \ g/cm^3$)

NOTE: $\sinh^{-1} z = \ln\left(z + \sqrt{1+z^2}\right)$

2B) PARTIAL DEGENERACY

$\frac{\rho}{\mu_e} > 2.4 \times 10^{-8}\, T^{3/2}\ \frac{g}{cm^3}$ dividing line between degeneracy & non-degeneracy is only for a gas of e^-'s. There is a gradual transition



as ρ increases, P(ε) takes the shape of a degenerate distribution gradually

Distribution of $\bar{e}$ momenta $n_e(p)\,dp = \frac{2}{h^3} 4\pi p^2 dp \frac{1}{e^{\alpha + E/kT} + 1}$

w/ α depending on $\bar{e}$ density & T.

To fix α $\longrightarrow$ $n_e = \int_0^\infty n_e(p)\,dp = n_e(\alpha, T)$

$$P_e = \frac{8\pi}{3h^3} \int_0^\infty \frac{p^3 v_p\, dp}{e^{\alpha + E/kT} + 1}$$ integral for the pressure of the perfect e^- gas

NOTE: for $T < 10^9$ K, non-relativistic $\bar{e}$ degeneracy sets in before relativistic degeneracy $\Rightarrow$ for partially degenerate gas we restrict to non-relativistic kinematics, ie: $v_p = p/m$

$$\Rightarrow P_e = \frac{8\pi}{3h^3 m} \int_0^\infty \frac{p^4\, dp}{e^{\alpha + p^2/2mkT} + 1} \quad \& \quad n_e = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2\, dp}{e^{\alpha + p^2/2mkT} + 1}$$

Changing variables $u = p^2/2mkT$ (dimensionless) $\rightarrow p = (2mkTu)^{1/2}$

$dp = \frac{mkT\, du}{(2mkTu)^{1/2}}$

$$\Rightarrow \begin{cases} P_e = \frac{8\pi kT}{3h^3} (2mkT)^{3/2} \int_0^\infty \frac{u^{3/2}\, du}{e^{\alpha + u} + 1} \equiv \frac{4\pi kT}{h^3} (2mkT)^{3/2} \frac{2}{3} F_{3/2}(\alpha) \\ n_e = \frac{4\pi}{h^3} (2mkT)^{3/2} \int_0^\infty \frac{u^{1/2}\, du}{e^{\alpha + u} + 1} \equiv \frac{4\pi}{h^3} (2mkT)^{3/2} F_{1/2}(\alpha) \end{cases}$$

w/ F Fermi-Dirac functions

NOTE: For $\alpha \to \infty$, $F_{3/2}/F_{1/2} \longrightarrow 3/2 \Rightarrow P_e \to n_e KT$, pressure of a maxwellian e^- gas.

For $\alpha \to -\infty$, $F_{3/2}/F_{1/2} \longrightarrow \frac{3}{5}\mu_0 \Rightarrow P_e \to \frac{8\pi}{15 m h^3} p_0^5$, pressure of a completely degenerate non-relativistic e^- gas.

$$\boxed{P_e = n_e KT \underbrace{\left(\frac{2}{3}\frac{F_{3/2}}{F_{1/2}}\right)}}$$

measure the extent to which the electron pressure differs from that of a non-degenerate gas.

For $\alpha > 2 \longrightarrow$ non degenerate

In terms of mass density, $n_e \equiv \frac{\rho}{\mu_e m_H} = \frac{4\pi}{h^3}(2mKT)^{3/2} F_{1/2}(\alpha)$

m: e^- mass

$$\boxed{\log\left(\frac{\rho}{\mu_e} T^{-3/2}\right) = \log F_{1/2}(\alpha) - 8.04} \quad (*)$$

NOTE: For a given (ρ, T), $(*)$ determines $F_{1/2}(\alpha)$, hence α, which in turn allows $F_{3/2}(\alpha)$ to be interpolated from the tables of the Fermi-Dirac functions

NOTE: In most stars, relativistic degeneracy is important only for such high densities that degeneracy is essentially complete.

Appropriate expansions of the equation of state that converge rapidly for weak degeneracy (nearly maxwellian) and for strong degeneracy (nearly complete).

WEAK NON-RELATIVISTIC DEGENERACY:

$\Lambda := e^{-\alpha} \longrightarrow \alpha > 0$ weak degeneracy, ie: $\Lambda < 1$

$$F_{1/2}(\Lambda) = \int_0^\infty \frac{u^{1/2}\,du}{(1/\Lambda)e^{u}+1} = \int_0^\infty \Lambda e^{-u} u^{1/2} \frac{1}{1+\Lambda e^{-u}}\,du \underset{\text{expanding}}{=}$$

$$= \Lambda \int_0^\infty e^{-u} u^{1/2}\left[1 - \Lambda e^{-u} + (\Lambda e^{-u})^2 - (\Lambda e^{-u})^3 + \dots\right] du$$

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\,du$$

$x = n + \frac{1}{2}$

$\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(n+1) = n\Gamma(n)$

$\Rightarrow$ eg. $\int_0^\infty u^{1/2} e^{-u}\,du = \Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$

Integrating term by term:

$$F_{1/2}(\Lambda) = -\frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}\frac{(-1)^n \Lambda^n}{n^{3/2}}, \quad \Lambda < 1$$

$\Updownarrow$ equivalently

$$\boxed{F_{1/2}(\alpha) = -\frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}\frac{(-1)^n e^{-n\alpha}}{n^{3/2}}, \quad \alpha > 0}$$

$$\Rightarrow n_e = \frac{4\pi}{h^3}(2mKT)^{3/2} F_{1/2}(\alpha) \underset{\alpha>0}{=} \frac{2}{h^3}(2\pi m KT)^{3/2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-n\alpha}}{n^{3/2}}$$

$$P_e = \frac{2KT(2\pi m KT)^{3/2}}{h^3}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-n\alpha}}{n^{5/2}}, \quad \alpha > 0$$

$$\Rightarrow P_e \underset{\alpha \text{ large enough}}{\simeq} n_e KT\left(1 + \frac{n_e h^3}{2^{7/2}(2\pi m KT)^{3/2}} + \dots\right) \simeq n_e KT\left(1 + 10^{-16.435} n_e T^{-3/2} + \dots\right)$$

STRONG NON-RELATIVISTIC DEGENERACY:

strong degeneracy when $\alpha \ll 0 \longleftrightarrow \lambda \gg 1$

LEMMA (from Chandrasekhar): If $\phi(u)$ is a sufficiently regular function which vanishes for $u=0$, then

$$\int_0^\infty \frac{du}{(1+\lambda)e^u+1}\frac{d\phi(u)}{du} = \phi(u_0) + 2\left[c_2\left(\frac{d^2\phi}{du^2}\right)_{u_0} + c_4\left(\frac{d^4\phi}{du^4}\right)_{u_0} + \ldots\right]$$

w/ $u_0 = \lg\lambda$ & $c_\nu = 1 - \frac{1}{2^\nu} + \frac{1}{3^\nu} - \frac{1}{4^\nu} + \ldots$

$\Rightarrow$ $$F_{1/2}(\alpha) = \frac{2}{3}(-\alpha)^{3/2}\left(1 + \frac{\pi^2}{8\alpha^2} + \frac{7\pi^4}{640\alpha^4} + \ldots\right)$$

$$F_{3/2}(\alpha) = \frac{2}{5}(-\alpha)^{5/2}\left(1 + \frac{5\pi^2}{8\alpha^2} - \frac{7\pi^4}{384\alpha^4} + \ldots\right)$$

good for $\alpha < -1$, accurate to three decimal places for $\alpha < -5.6$ & useful for $\alpha < -3$.

Since $n_e = \frac{4\pi}{h^3}(2mKT)^{3/2}F_{1/2}(\alpha) \Rightarrow -\alpha \approx \frac{1}{2mKT}\left(\frac{3h^3 n_e}{8\pi}\right)^{3/2}$

in case of strong degeneracy

$$p_0 = \left(\frac{3h^3}{8\pi}n_e\right)^{1/3}$$

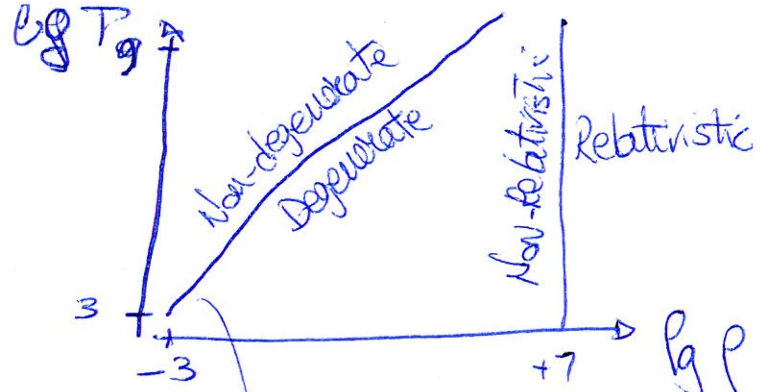
$$E_f = \frac{1}{2}\frac{p_0^2}{m} \quad \text{Fermi energy}$$

$\Rightarrow$ $-\alpha \approx \frac{p_0^2}{2mKT} = \frac{E_f}{KT}$

w/ E_f Fermi energy (the Kinetic energy of an electron at the top of the Fermi sea)

$\Rightarrow$ For $\alpha < -3$, $$\frac{\rho}{\mu_e}T^{-3/2} \approx \frac{2}{3}(-\alpha)^{3/2}\left(1 + \frac{\pi^2}{8\alpha^2} + \frac{7\pi^4}{640\alpha^4} + \ldots\right)10^{-8.04}$$

Relating α & the density-temperature.



Eq. of state of the perfect e^-s gas.

$$\lg T \propto \frac{2}{3} \lg \rho$$

slope

$$\frac{\rho}{\mu_e} = 2.4\times10^{-8}\, T^{3/2}\ \frac{g}{cm^3}$$: around this boundary, the equation of state needs to be evaluated w/ $\lg \frac{\rho}{\mu_e} T^{-3/2} = \lg F_{1/2}(\alpha) - 8.04$, which applies to partial degeneracy.

$$P_e = \frac{8\pi KT}{3h^3}(2mKT)^{3/2} F_{3/2}(\alpha)$$

For densities as high as $\frac{\rho}{\mu_e} = 7.3\times10^{6}\ \frac{g}{cm^3}$ ⟹ the e^-s gas becomes relativistic (vertical line).

Around this boundary, the eq. of state can be evaluated w/ $P_e = \frac{\pi m^4 c^5}{3h^3} f(x)$

For very high temperatures ($T > 10^9$ K), the e^- gas can be both relativistic & only partially degenerate.

Regarding the mechanical pressure supporting the star ($\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$, eq. of hydrostatic equilibrium), $P = P_e + P_{nuclei}$. Nuclei are never degenerate in common stars (exception, neutron stars), P_{nuclei} is that of a maxwellian gas, using the appropriate value of μ. Since e^-s have been accounted separately, use only the μ of the remaining ions & nuclei.

⟹ $P_{gas} = P_e + \frac{\rho KT}{\mu_i m_H}$, μ_i: mean molecular weight of ions.

In practical cases where e^- degeneracy occurs, nuclei are generally He, C, O nuclei, & perhaps even heavier nuclei $\Rightarrow$ P_e provides the bulk of the pressure, w/ P_i only a small addition.

NOTE: For weak non-relativistic degeneracy

$$P_e \simeq n_e KT\left[1 + \frac{n_e h^3}{2^{7/2}(2m\pi KT)^{3/2}}\right] = n_e KT[1+D]$$

correcting factor

$$\text{w/ } D \propto \frac{n_e}{m^{3/2}\,T^{3/2}}$$

For fixed n & T $\Rightarrow$ $\dfrac{D_{\text{nuclei}\,(n,p)}}{D_e} = \left(\dfrac{m_e}{m_p}\right)^{3/2} \simeq 10^{-5}$

$\Rightarrow$ when the density increases, D becomes non-negligible for e^-s well before than ps & ns.

The pressure of a completely degenerate gas does not explicitly depend on T. When $E_f = \frac{1}{2}\frac{p_0^2}{m} \gg KT \Rightarrow$ complete degeneracy. In this case, P_e is approximately independent of the T.

$\Rightarrow$ a small rise in T of an almost complete degenerate e^- gas causes almost no change in the pressure.

This is very crucial on stellar structures & stellar evolution (runaways in nuclear reaction rates, aka flash phenomena).

The normal processes of energy transport in stellar interiors are altered when the e^- gas is degenerate, i.e.: <u>heat conductivity</u> becomes important. When non-degenerate, the m.f.p. of charged particles is so small that conductivity is extremely inefficient. But when degenerate, the m.f.p. of e^-s is quite long. The filling up of available states in momentum space below a certain level prevents energy exchange among e^-s, making energetic e^-s free to move about $\Rightarrow$ partially degenerate e^- gases isothermal. (2-8)

WHITE DWARFS are supported by completely degenerate e^- gas. As they cool radiating thermal energy, the nearly degenerate momentum distribution becomes increasingly rectangular

NOTE: The temperature no longer corresponds to kinetic energy in a degenerate gas.

PHOTON GAS & RADIATION PRESSURE

Pressure is also extorted by the radiation field in the interior of stars.
Each quantum of elm energy $h\nu$ carries a momentum $p_\gamma = h\nu/c$
If thermodynamic equilibrium $\Rightarrow$ radiation flux is isotropic

$\Rightarrow$ pressure integral $P = \frac{1}{3}\int_0^\infty p\, v_p\, n(p)\,dp = \frac{1}{3}\int_0^\infty \frac{h\nu}{c}\, c\, n(\nu)\,d\nu = \frac{1}{3}\underbrace{\int_0^\infty h\nu\, n(\nu)\,d\nu}_{\text{energy density of photons, } u}$

$$\Rightarrow \boxed{P_r = \frac{1}{3}u = \frac{1}{3}aT^4} \qquad a = 7.565 \times 10^{-15} \frac{\text{erg}}{\text{cm}^3\,{}^\circ\text{K}^4}$$

$u = aT^4$

In stars, there is a net excess of radiation energy flowing in one particular direction $\Rightarrow$ the radiation field is slightly anisotropic.
For cases appropriate to the interiors of stars, where near thermodynamic equilibrium, the radiation field may be approximated by

$$I(\theta) = \underbrace{I_0}_{\text{isotropic part}} + \underbrace{I_1\cos\theta}_{\text{anisotropy in the radiation field corresponding to the net flux in the polar direction}} + \ldots$$

$u := \frac{1}{c}\int I(\theta)\,d\Omega = \frac{2\pi}{c}\int_0^\pi I(\theta)\sin\theta\,d\theta$

$\Rightarrow u = \frac{4\pi}{c} I_0$ independent of the anisotropic term in the radiation field

$P_r = \frac{4\pi}{3c} I_0 = \frac{a}{3}T^4$, same as for the isotropic case.

$P_r := \frac{1}{c}\int I(\theta)\cos^2\theta\,d\Omega = \frac{2\pi}{c}\int_0^\pi I(\theta)\cos^2\theta\sin\theta\,d\theta$

Net flux of energy in the polar direction $H := \int I(\theta)\cos\theta\,d\Omega = \frac{4\pi}{3} I_1$

dependent on the existence of the anisotropic term.

$= 2\pi\int_0^\pi I(\theta)\cos\theta\sin\theta\,d\theta$

$$\Rightarrow I(\theta) \approx \frac{c}{4\pi}u + \underbrace{\frac{3}{4\pi}H\cos\theta}_{\ll \frac{c}{4\pi}u \text{ in the interior of stars}}$$

For a black body radiation, the energy emitted per unit of area from the surface

$$J = 2\pi \int_0^{\pi/2} (I_0 + I_1 \cos\theta) \cos\theta \sin\theta \, d\theta = \pi I_0 + \frac{2\pi}{3} I_1$$

For an isotropic radiation field (or when $\pi I_0 \gg \frac{2\pi}{3} I_1$)

$$\Rightarrow \boxed{J = \pi I_0 = \frac{c}{4} u = \sigma T^4}$$

Stefan-Boltzmann constant $\sigma = \frac{c}{4} a$,

$$\sigma = 5.67 \times 10^{-5} \frac{\text{erg}}{\text{cm}^2 \, \text{s} \, K^4}$$

The mechanical pressure of a perfect gas

$\Rightarrow P = P_{ions} + P_{electrons} + P_{radiation}$ (valid for $T < 10^9 K$)

If e^- gas non degenerate $\Rightarrow P_{ions} + P_{e^-} = P_g = \frac{\rho K T}{\mu m_H}$

w/ μ mean molecular weight of all free particles

If e^- gas degenerate $\Rightarrow P_{ions} = \frac{\rho K T}{\mu_{ions} m_H}$, w/ μ_i mean molecular weight of ions

$P_e \propto \begin{cases} (\rho/\mu_e)^{5/3} & \text{completely deg. non-relativ.} \\ (\rho/\mu_e)^{4/3} & \text{relativistic} \end{cases}$ ← P_e = appropriate equations for degenerate e^- pressure

For $T > 10^9 K$ (when positron-e^- pairs may be produced from energetic photons) or for very high densities (where particle interactions may invalidate the perfect gas approximation)

$\Rightarrow P \neq P_{ions} + P_e + P_{rad}$.

Also $P_{magnetic}$ if magnetic fields are present in low-density regions.

For a perfect, nondegenerate gas, $P_g = P_r$ when

$$\frac{\rho K T}{\mu m_H} = \frac{1}{3} a T^4 \underset{eq.}{\Leftrightarrow} T = 3.2\times10^7 \left(\frac{\rho}{\mu}\right)^{1/3} \simeq 3.6\times10^7 \rho^{1/3}$$

$$\lg T \simeq \underbrace{\frac{1}{3}}_{\text{slope}} \lg \rho + 7.56$$

← for nearly all H