STELLAR STRUCTURE

by Danilo Marchesini

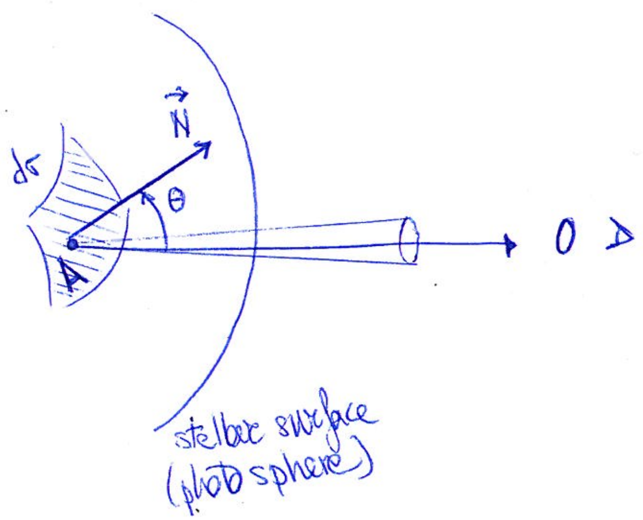
Total power output of a star $L = L_\gamma + L_\nu + L_m$

photon luminosity ($[L] = erg/s$)

neutrino luminosity

mass-loss luminosity

Let's focus on the photon luminosity L_γ. This originates from the photosphere of stars (i.e., the surface). From the interior of a star I get no photons, due to the total opacity, except neutrinos.



$\vec{AO}$ direction of observer

$\vec{N}$ normal vector to $d\sigma$

$d\sigma$ surface element of photosphere

θ angle between $\vec{N}$ & $\vec{AO}$

Energy emitted by $d\sigma$ in the unit of time, in the solid angle $d\omega$ around the direction $\vec{AO}$ in the frequency interval $d\nu$

$$dE_\nu = I_\nu(\theta) \cos\theta \, d\sigma \, d\nu \, d\omega$$

$\cos\theta$: to have the component along $\vec{AO}$

$I_\nu(\theta)$: specific intensity of the radiation

$[I_\nu(\theta)] = erg/s/cm^2/str/Hz$

Energy flux $F_\nu = \int I_\nu(\theta) \cos\theta \, d\omega$

In the assumption of spherical symmetry $d\omega = 2\pi \, \mathrm{sen}\,\theta \, d\theta$

NOTE: $d\omega = \sin\theta d\theta d\phi$
$\phi \in [0, 2\pi]$
$\theta \in [0, \pi/2]$ semi-sphere

$$\Rightarrow \quad F_\nu = 2\pi \int_0^{\pi/2} I_\nu(\theta) \, \mathrm{sen}\,\theta \cos\theta \, d\theta$$

integrating for the outward hemisphere

F_ν is the total energy emitted in the unit of time, unit of surface & unit of frequency, ie: $[F_\nu] = \mathrm{erg/s/cm^2/Hz}$ (or energy flux)

NOTE: $\nu = c/\lambda \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2} \longrightarrow F_\lambda = \frac{c}{\lambda^2} F_\nu$

I am interested in the energy emitted by the photosphere in the unit of time $\longrightarrow$ integral over surface

$$L_\nu = \underbrace{4\pi R^2}_{\text{surface of sphere}} F_\nu \quad , \quad \text{w/ } L_\nu \text{ monochromatic photon luminosity}$$

$[L_\nu] = \mathrm{erg/s/Hz}$

NOTE: A star is always seen as a pointlike object (except the Sun & a handful of other cases). Therefore, observations always return L_ν (or F_ν if I know the radius of the star). Only for the Sun (& a few other stars) I am able to determine I_ν.

Integrating over all frequencies, we obtain the total energy emitted by the star in the unit of time, or total luminosity L, $[L] = \mathrm{erg/s}$

$$L = 4\pi R^2 \int_0^\infty F_\nu \, d\nu = 4\pi R^2 \int_0^\infty F_\lambda \, d\lambda .$$

photon luminosity (ie:; L_γ)

What is the amount of emitted energy available to the observer?

If the star is at a distance r from the instrument:

$$f_\nu = \frac{L_\nu}{4\pi r^2} = \frac{R^2}{r^2} F_\nu \qquad \left(\text{or equivalently, } f_\lambda = \left(\frac{R}{r}\right)^2 F_\lambda\right)$$

substituting $L_\nu = 4\pi R^2 F_\nu$

f_ν is the incident energy flux

w/ $[f_\nu]$ = erg/s/cm^2/Hz

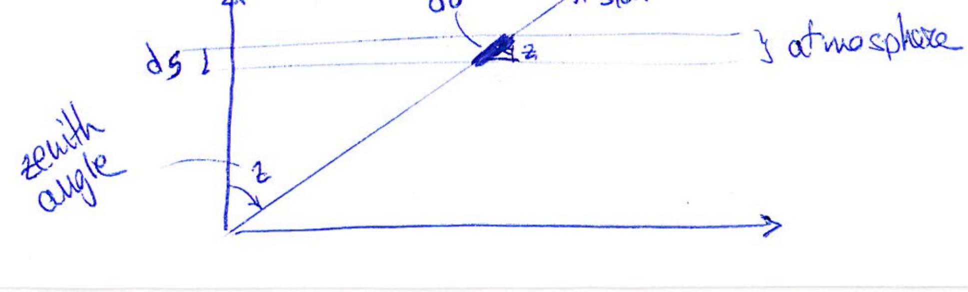
NOTE: 1) This is true only in the absence of interstellar absorption.
2) From a measurement of $r^2 f_\nu$, we're able to derive $R^2 F_\nu$

Let's now consider a detector proportional to the signal, that is its response $l_\nu \propto f_\nu$. Note that the photons can be further affected during their travels to the observer.

Causes of absorption:

1. INTERSTELLAR ABSORPTION: the interstellar material is made of gas & dust (very low density), not uniformly distributed (more in the disk). The absorption is a function of position of the source, its distance, i.e., & of frequency.
This absorption is defined by the parameter $\alpha_\nu \lesssim 1$ ($\alpha_\nu \simeq 1$ if sources close to the Sun).

2. ATMOSPHERIC ABSORPTION: The Earth's atmosphere always attenuates radiation. It is completely opaque at certain wavelengths, but it is never transparent.



The thickness of the atmosphere is much smaller than r
→ azimuthal distance z only parameter

Let's consider a radiation w/ specific intensity I_ν going (4)
through the atmosphere. The section crossed is
$da = ds \sec z$ $\left(\sec\alpha = \frac{1}{\cos\alpha}\right)$

The attenuated intensity is $dI_\nu = -\underbrace{\chi_\nu}_{\text{coefficient of absorption}} I_\nu da = -\chi_\nu I_\nu ds \sec z$

$$\Rightarrow \frac{dI_\nu}{I_\nu} = -\chi_\nu \sec z \, ds$$

Integrating $I_\nu^{obs} = I_\nu^{int} e^{-K_\nu \sec z}$

this term includes the effects of diffusion as well.

$$\Rightarrow I_\nu^{obs} = I_\nu^{int} a_\nu, \quad w/\ a_\nu = e^{-K_\nu \sec z} = A_\nu e^{-\sec z}$$

$A_\nu :=$ coefficient of extinction
For a specific place, A_ν is known.
Measuring z & knowing A_ν $\Rightarrow$ all astronomical observations are effectively converted to zenith, ie: as all sources were at zenith.

3. Absorption effects du to optics in instruments : Q_ν factor
4. Detector response : S_ν (assuming the response proportional to f_ν)
In S_ν I also include the filter passbands.

$$\Rightarrow b_\lambda = \underbrace{\Sigma}_{\text{collecting area of telescope}} \cdot a_\nu \cdot A_\nu \cdot Q_\nu \cdot S_\nu \cdot f_\nu$$

effective energy I measure

$[b_\lambda] = erg/s/Hz$

$\Sigma = \pi a^2$

Substituting $P_\nu = A_\nu Q_\nu S_\nu$ (measurable once for all accurately)
& a telescope of unit area

$$\Rightarrow \boxed{l_\nu = a_\nu P_\nu f_\nu}$$

apparent flux
$[l_\nu] = erg/s/cm^2/Hz$

Measuring ℓ_ν, I can derive f_ν once I know P_ν & the ISM extinction α_ν. If the distance of the source is known (r), I know L_ν & F_ν.

Since filters are used in photometry

$$\Rightarrow \quad \ell = \int_0^\infty \alpha_\nu P_\nu f_\nu \, d\nu \qquad [\ell] = erg/s/cm^2$$

Apparent brightness

The human eye has a nearly logarithmic subjective response to radiant energy flux (Pogson's formula)

$$\underbrace{m_2 - m_1}_{\text{apparent magnitudes of two stars}} = 2.5 \log \frac{\ell_1}{\ell_2}$$

ℓ_1, ℓ_2: corresponding apparent brightness of the two stars

$$\Rightarrow \quad \text{Apparent magnitude} \equiv m := -2.5 \log \ell + c$$

c: zero-point constant that depends on the specif system of magnitudes used & the units of the brightness.

Photometric systems of magnitudes

- Photographic system: $m_{pg} = m_v$ (v: visual) for A0 stars
- Photoelectric systems: example UBV Johnson & Morgan
  (Vega system) Zero point determined using the sequence of standard stars.

  $U - B = B - V = 0$ for A0 V (Vega, T=10000 K) "V" Main sequence, "A" spectral type.

  $\lambda_U \approx 3650$ Å $\quad \lambda_B \approx 4400$ Å similar to photographic plates $\quad \lambda_V \approx 5500$ Å similar to visual band

- Spectro-photometric system: $\lambda_{eff} := \dfrac{\int_0^\infty \lambda P_\lambda \, d\lambda}{\int_0^\infty P_\lambda \, d\lambda}$ effective wavelength $\nu_{eff} = c/\lambda_{eff}$

- AB system: $m_{AB} := -2.5 \log \ell_\nu - 48.6$ w/ $[\ell_\nu] = erg/s/cm^2/Hz$

The apparent magnitudes do not represent the intrinsic luminosity of a star. For an absolute comparison of intrinsic brightness it is common to discuss the magnitudes of stars would have if they were all at the same geocentric distance.

ABSOLUTE MAGNITUDE of a star defined as its magnitude viewed from a distance of 10 pc (M)

Assuming no absorption $\alpha_\nu = 1$, $m_\nu = -2.5 \log f_\nu + c$ (monochromatic magnitude)

$$f_\nu = \frac{L_\nu}{4\pi r^2} \Rightarrow m_\nu = -2.5 \log \frac{L_\nu}{4\pi r^2} + c$$

$$M_\nu = -2.5 \log \frac{L_\nu}{4\pi (10\,pc)^2} + c$$

$$\Rightarrow M_\nu - m_\nu = -2.5 \log \frac{L_\nu}{4\pi (10\,pc)^2} \cancel{+c} + 2.5 \log \frac{L_\nu}{4\pi r^2} \cancel{-c} =$$

$$= \cancel{-2.5 \log L_\nu} + \cancel{2.5 \log (4\pi)} + 2.5 \log (10\,pc)^2 \cancel{+2.5 \log L_\nu} + \cancel{-2.5 \log 4\pi} - 2.5 \log r^2 =$$

$$= 5 \log (10\,pc) - 5 \log r$$

$$\Rightarrow \boxed{M_\nu - m_\nu = 5 - 5 \log r\,[pc]}$$

$m_\nu - M_\nu$ = distance modulus

$$m_\nu - M_\nu = 5 \log r - 5$$

what we are really interested in is the physical quantity of the rate at which photon energy is radiated from the star (erg/s), or luminosity.

Luminosity is measured by the ABSOLUTE BOLOMETRIC MAGNITUDE

$$L = 4\pi R^2 \int_0^\infty F_\nu \, d\nu \qquad m_{bol} = -2.5 \log \int_0^\infty f_\nu \, d\nu + c$$

$[L]$ = erg/s

w/ $\int_0^\infty f_\nu \, d\nu = \frac{L}{4\pi r^2}$

$$M_{bol} := -2.5 \log L + c$$

$$\Rightarrow \quad M_{bol} - m_{bol} = 5 - 5 \log r[pc]$$

$$\boxed{M_{bol} - M_{bol,\odot} = -2.5 \log \frac{L}{L_\odot}}$$

$$M_{bol,\odot} = 4.72 \Rightarrow \boxed{\begin{array}{l} M_{bol} = -2.5 \log \frac{L}{L_\odot} + 4.72 \\ L_\odot \cong \underset{(\pm 0.04)}{3.90} \times 10^{33} \text{ erg/s} \end{array}}$$

M_{bol} is not directly measurable. To obtain the total energy radiated from a star requires making a BOLOMETRIC CORRECTION.

SUMMARY: Astronomers measure the apparent magnitude m & distance of a star. The absolute magnitude M is computed by mentally viewing it a r = 10 pc. The M is then converted to an absolute bolometric magnitude using BC.

$$M_{bol} = M_V - BC$$

$$M_{bol} = -2.5 \log L + c = -2.5 \log L[\text{erg/s}] + 88.70$$

$$= -2.5 \log \frac{L}{L_\odot} + 4.72$$

NOTE: $BC := 2.5 \log \dfrac{\text{incident energy flux}}{\text{recorded energy flux}} = 2.5 \log \dfrac{\int f_\nu d\nu}{\int \alpha_\nu P_\nu f_\nu d\nu}$

BC are tabulated as a function of the different spectral type stars. Known the spectral type, I can derive it's absolute bolometric magnitude M_{bol} once I know BC & M_V.

$$M_V|_\odot = 4.83$$

Neutrino Luminosity L_ν

- mean free path of a neutrino in water is $10^9 * R_\odot$!
  → neutrinos ν_s produced in the star interior escape from it w/o interaction (only exception is for extreme T or ρ, as in imploded cores of SN).

- L_ν is not observed, not determined from observations, but must be calculated from the model of the star.

Mass-Loss Luminosity $L_{\dot m}$

* catastrophic mass loss (e.g. supernova, w/ masses comparable to that of the sun being blasted off w/ $v \sim$ thousands of km/s)
  ↳ rapid & violent change in the structure of the star.

* Planetary nebula, red giants & T-Tauri

$\underbrace{\text{red giants \& T-Tauri}}$ matters flowing w/ $v \sim$ tens - hundreds km/s

in amounts as great as $10^{-7} M_\odot$/yr.

* For the vast majority of stars, escaping matter cannot be seen @ all (eg., corona for the Sun).
stellar winds

$$L_{\dot m} = \left[G\frac{M}{R} + (U_\infty - U_e) + \frac{1}{2} v_\infty^2 \right] \frac{dM}{dt}$$

rate at which mass crosses an appropriate circumstellar sphere

$\frac{1}{2} v_\infty^2$: velocity at large distances (when unbound)

$(U_\infty - U_e)$: internal energy per unit of mass at the photosphere (U_e) & at very large distances (U_∞)

For ionized matter $U = \frac{3}{2} kT * N$

N: # of particles per unit of mass

$G\frac{M}{R}$: $\Omega_G|_e = -\frac{GM}{R}$ gravitational potential energy at the photosphere

$\Omega_G|_\infty = 0$

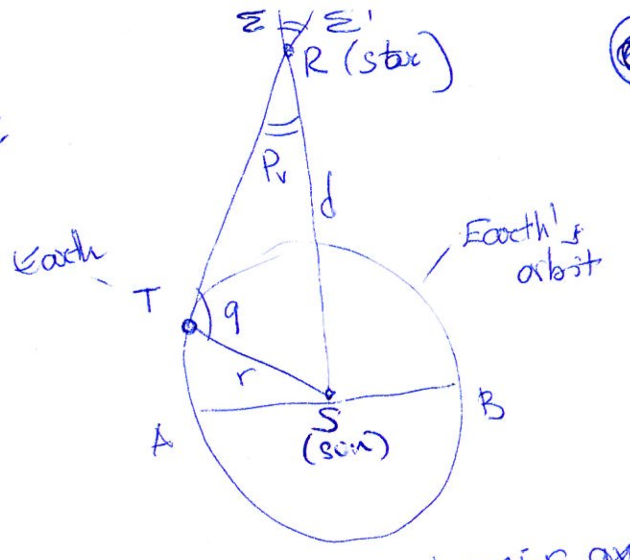
$L_{\dot{m}}\big|_{\odot} \sim 2\times10^{-6}\,L_\odot \lll L_\odot$ negligible

NOTE: For the Sun, the mass is pushed away hydrodynamically ...

The necessary power for mass loss must be derived from the body of the star.

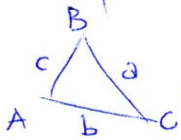
Parallax ; $\hat{\Sigma\Sigma'} = T\hat{R}S \equiv P_r$ (9b)

annual parallax of the star (variable)



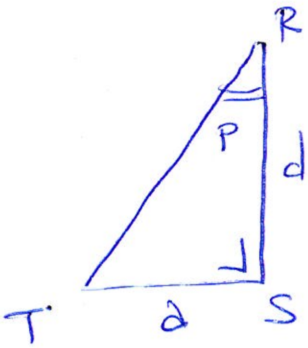
$r = \overline{TS}$ $d = \overline{RS}$ $q = S\hat{T}R$

$\Rightarrow \operatorname{sen} P_r = \frac{r}{d} \operatorname{sen} q$



$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

AS = semi-major axis earth's orbit.



I define the angle P such that $\operatorname{tg} P = \frac{a}{d}$

i.e.: P is the angle corresponding to the semi-major axis of the Earth's orbit if observed from the star R.

$\Rightarrow$ P parallax angle $\equiv \max(P_r)$

Since P is small for all stars $\Rightarrow \operatorname{tg} P \approx P$

$\Rightarrow P = a/d = \frac{a}{d} R'' = \frac{a}{d} 206265$

if $[P] =$ arcsec

$a = 1\,AU = 1.496 \times 10^8$ km

$a \cdot 206265 = 3.086 \times 10^{13}$ km $:= 1$ pc

$$\Rightarrow \boxed{d\,[pc] = \frac{1}{p\,['']}}$$

Closest star, α-Centauri : $P = 0.76''$

$\frac{\Delta d}{d} = \frac{\Delta p}{p}$ $\Delta p \lesssim 35\,\% \longleftrightarrow d \lesssim 50$ pc

From the ground, error $\Delta p \sim 0.007''$

$\Rightarrow$ @ $d = 50$ pc , $\Delta d \simeq 35\%$

INTERSTELLAR ABSORPTION

Let's consider the monochromatic magnitude

$$m_\nu = -2.5 \lg f_\nu^* + c \quad , \quad w/ \quad f_\nu^* = \alpha_\nu f_\nu = \alpha_\nu \frac{L_\nu}{4\pi r^2}$$

$$f_\nu = \frac{L_\nu}{4\pi r^2} \quad , \; r \text{ distance}$$

$$\Rightarrow m_\lambda = \underbrace{-2.5 \lg f_\lambda}_{m_\lambda^\circ} \underbrace{- 2.5 \lg \alpha_\lambda}_{\text{interstellar absorption}} + c$$

$$= m_\lambda^\circ - 2.5 \lg \alpha_\lambda \quad \Rightarrow \quad A_\lambda \equiv (m_\lambda - m_\lambda^\circ)$$

Monochromatic absolute magnitude

$$M_\lambda = m_{o\lambda} + 5 - 5 \lg r = m_\lambda + 5 - 5 \lg r - A(\lambda)$$

$$w/ \quad A(\lambda) = -2.5 \lg \alpha_\lambda$$

ie: I treat the absorption as a magnitude term.

Assumption: the properties of the absorbing matter are equal everyway in the galaxy $\Rightarrow A(\lambda) = f(r)\, f(\lambda)$

term depending on the position of the star

term only depending on the wavelength

I call $K(\lambda)$ coefficient of absorption (independent on the location) such that the radiation is absorbed by $\rho K(\lambda)\, dl$ when it travels through a cylinder of density ρ & length dl & unitary section

dl

$$\Rightarrow \alpha_\lambda = e^{-K(\lambda)\int_0^r \rho\, dl}$$

$[K(\lambda)] = \frac{cm^2}{g}$ cross section for absorbing photons of wavelength λ per unit of mass.

$\Rightarrow A_\lambda = -2.5 \lg \alpha_\lambda = -2.5 \left[-K(\lambda) \int_0^r \rho \, dl \right] \lg e$

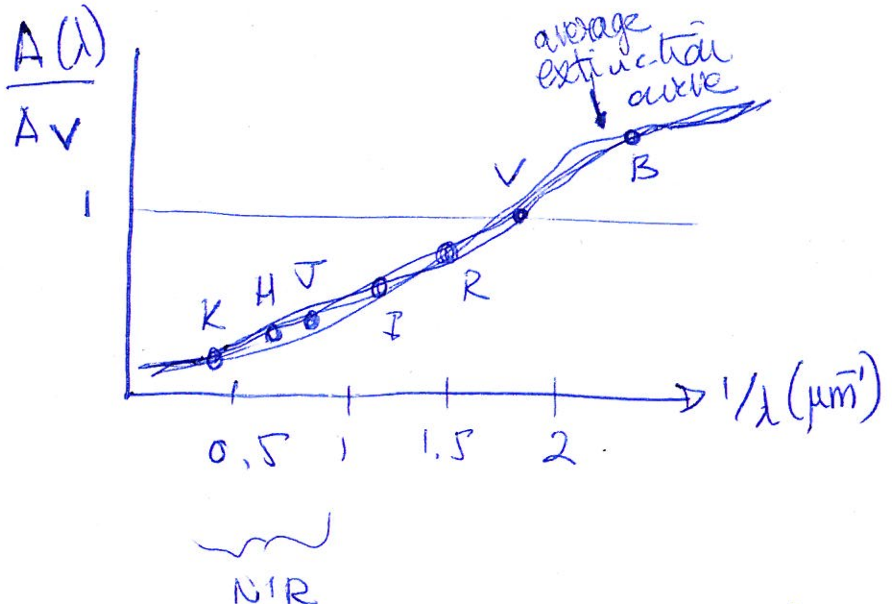
$\Rightarrow A_\lambda = \mathbb{K} \cdot K(\lambda)$ w/ $\mathbb{K} = 2.5 \lg e \int_0^r \rho \, dl = 1.086 \int_0^r \rho \, dl$

$\Rightarrow A_\lambda = 1.086 \underbrace{K(\lambda) \int_0^r \rho \, dl}_{\tau_\lambda := \text{optical depth}} = 1.086 \tau_\lambda$

NOTE: $\mathbb{K}$ depends only on r (ie: it is the term $f(r)$) & on how the interstellar matter is distributed

$\int_0^r \rho \, dl :=$ mass contained in the cylinder of unitary section between the observer & the star.

NOTE: $K(\lambda)$ does not depend on direction except for some special cases (it is however different in different galaxies).



$A(\lambda) \to 0$ for increasing λs ie: to make observations in regions w/ lots of dust (for example, MW center), it best to observe at larger wavelengths, in the NIR.

$A(\lambda) \propto 1/\lambda$ in $1 \lesssim \frac{1}{\lambda} \lesssim 2$

Let's consider two wavelengths, λ_1 & λ_2.

$\Rightarrow \underbrace{m_2 - m_1}_{\text{observed color index}} = \underbrace{m_2^0 - m_1^0}_{\text{intrinsic color index (before extinction)}} + \underbrace{E(\lambda_2 - \lambda_1)}_{\text{excess color}}$

w/ $E(\lambda_2 - \lambda_1) := A(\lambda_2) - A(\lambda_1)$ effect of the ISM absorption on the color index

Since $\lambda_1 > \lambda_2 \Rightarrow E(\lambda_2 - \lambda_1) > 0$

$\Rightarrow$ REDDENING DUE TO ISM ABSORPTION

The curve A_λ / A_V (or A_λ / A_J) is called the extinction curve or law for a specific line of sight.

A_λ peaks in the far-UV; shorter wavelength radiation (X-rays) passes right through grains, while much larger wavelength photons refracts around them. At very long wavelengths ($\lambda \sim 100\,\mu m$), $A_\lambda \propto 1/\lambda$. In most the OPT & UV regions, A_λ is falling slightly more steeply, w/ this steady fall broken by three distinct bumps.

UV bump @ 2170 Å (due to tiny pieces of graphite w/ ~50 atoms)
IR bumps @ 9.7 μm & 18 μm (associated w/ Si–O bands in silicate grains).

In the visual band & shorter λs, A_λ vary from one line of sight to another, perhaps due to variations of the size distribution of grains in different regions.

Slope of extinction curve near the V band is $A_V / A_J R_V$

$$\text{w/ } R_V := \frac{A_V}{A_B - A_V} = \frac{A_V}{E(B-V)}$$

which quantifies how steeply the extinction curve is rising into the UV.

$R_V \simeq 3 \rightarrow$ steeply rising
$R_V \simeq 5 \rightarrow$ slowly rising

In the MW, $\boxed{R_V \simeq 3.1} \longrightarrow A_V$ from readily measured quantity $E(B-V)$

$$\frac{A_\lambda}{A_V} = a(x) + \frac{b(x)}{R_V} \quad \text{w/ } x = 1/\lambda$$

(Cardelli, Clayton & Mathis 1989)

STELLAR TEMPERATURES:

For a gas, the most probable configuration depends upon the nature of the gas particles: (ie: the configuration of thermal equilibrium)

1. classical limit, ie: identical but distinguishable particles
2. identical, but indistinguishable particles of half-integral spin angular momentum (e^-s, ps, ns)
3. identical, but indistinguishable particles of integral spin angular momentum (γs, He^4 nuclei, π mesons)

ε energy of the gas particle

$n(\varepsilon)$ number density of gas particles of energy ε

$g(\varepsilon)$ # of possible particle states of energy ε

1) $$n(\varepsilon) = \frac{g(\varepsilon)}{e^{\alpha + \varepsilon/KT} + 0}$$ Maxwell-Bolzmann statistics

2) $$n(\varepsilon) = \frac{g(\varepsilon)}{e^{\alpha + \varepsilon/KT} + 1}$$ Fermi-Dirac statistics

3) $$n(\varepsilon) = \frac{g(\varepsilon)}{e^{\alpha + \varepsilon/KT} - 1}$$ Einstein-Bose statistics

most probable configurations

$\alpha \in \mathbb{R}$, depending on density of particles $\begin{cases} \alpha \gg 0 & \text{for small densities} \\ \alpha \ll 0 & \text{for very high densities} \end{cases}$

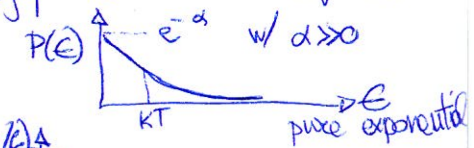
Total particle density := $\int n(\varepsilon)\, d\varepsilon$

For photons (zero-mass bosons) $$n(\varepsilon) = \frac{g(\varepsilon)}{e^{\varepsilon/KT} - 1}$$

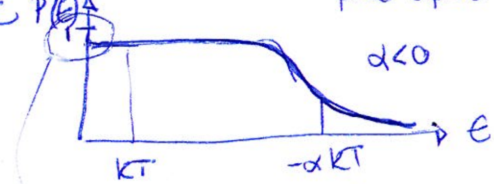
(ie: $\alpha = 0$)

$P(\varepsilon) := n(\varepsilon)/g(\varepsilon)$ OCCUPATION INDEX, i.e. probability that a state of energy ϵ is occupied. (11)

$\Rightarrow$ 1) $P(\epsilon) = e^{-\alpha} e^{-\epsilon/KT}$ Maxwell-Boltzmann

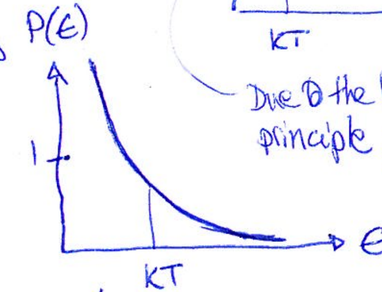


2) $P(\epsilon) = \dfrac{1}{e^{\alpha+\epsilon/KT}+1}$ Fermi-Dirac



Due to the Pauli exclusion principle for fermions

3) $P(\epsilon) = \dfrac{1}{e^{\epsilon/KT}-1}$ photons



large occupation index for $\epsilon \ll KT$ and an exponentially decreasing index for $\epsilon \gg KT$

<u>NOTES:</u> when $\alpha \gg 0$, $P(\epsilon)\Big|_{\text{Fermi Dirac}} \rightsquigarrow P(\epsilon)\Big|_{\text{Maxwell Boltzmann}}$

$P(\epsilon)$ Fermi-Dirac used to compute the pressure of gas made of e^-s

The number of gas particles in each free state of given energy

$n(\varepsilon) = P(\epsilon) \cdot g(\epsilon)\, d\epsilon$

$g(\epsilon)$ density of free states, $g(\epsilon)d\epsilon$ = # of states per unit volume of energy ϵ in the range $d\epsilon$

Note that particles can possess internal degrees of freedom that increase the total density of states.

For photons & electrons, the density of states may be written as $g(p)\,dp = \dfrac{2}{h^3} 4\pi p^2 dp$ (in momentum space)

EXERCISE 1: calculate the density of free electrons w/ energy E in the range dE, assuming that electrons are non-relativistic

SOLUTION: $n(E) = P(E)\, g(E)\, dE$

w/ $P(E) = \dfrac{1}{e^{\alpha + E/KT} + 1}$ $\qquad g(p)\,dp = \dfrac{2}{h^3} 4\pi p^2 dp$

$E = p^2/2m \longrightarrow p = \sqrt{2mE}$

$dp = \sqrt{\dfrac{m}{2}}\, E^{-1/2}$

$$g(p)\,dp = \frac{2}{h^3} 4\pi p^2 dp = \frac{8\pi}{h^3} 2m_e E \sqrt{\frac{m_e}{2}}\, E^{-1/2} = \frac{8\pi}{h^3} \left(2m_e^3\right)^{1/2} E^{1/2}$$

$$\Rightarrow n(E) = \frac{8\pi}{h^3} \left(2m_e^3\right)^{1/2} \frac{E^{1/2}\, dE}{e^{\alpha + E/KT} + 1}$$

NOTE: α is determined using $\int n(E)\,dE = n_e$ total density of free e^-'s

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At the low density of the outer layers of stars $\alpha >>> 1$

$\Rightarrow e^{\alpha + E/KT} + 1 \simeq e^{\alpha + E/KT}$, ie: unimportance of the Fermi-Dirac nature of e^-'s @ low density $\Rightarrow$ Maxwell-Boltzmann statistics

$$n_e(p)\,dp = A\, e^{-p^2/2mKT}\, 4\pi p^2 dp$$

, w/ A normalization constant from $\int n_e(p)\,dp = n_e$.

EXERCISE 2: After performing $\int n_e(p)\,dp = n_e$, show that

$$n_e(p)\,dp = \frac{4\pi p^2 dp}{(2\pi m_e KT)^{3/2}}\, n_e\, e^{-p^2/2m_e KT}$$

Maxwell Boltzmann distribution of e^- momenta in thermal equil.

$$n_e(E)\,dE = \frac{2\pi\, n_e}{(\pi KT)^{3/2}}\, E^{1/2} e^{-E/KT}\, dE$$

Maxwell-Boltzmann distribution of e^- energies in thermal equilibrium.

EXCERCISE 3: Using $n(E) = \dfrac{g(E)}{e^{E/KT}-1}$ ($\alpha = 0$ for photons)

& $p_\nu = h\nu/c$, derive the energy density of photons of frequency ν in the range $d\nu$ in thermal equilibrium.

SOLUTION: $E_\gamma = h\nu$

Energy density $u(\nu)\,d\nu = E(\nu)\,n(\nu)\,d\nu = h\nu\, n(\nu)\,d\nu$

$$n(\nu)\,d\nu = \underbrace{\frac{g(\nu)}{e^{h\nu/KT}-1}}_{E=h\nu} d\nu = \frac{8\pi}{c^3}\nu^2 d\nu \frac{1}{e^{h\nu/KT}-1}$$

$$g(\nu)\,d\nu = g(p)\,dp = \frac{2}{h^3}4\pi p^2 dp \underset{\substack{p=h\nu/c \\ dp=\frac{h}{c}d\nu}}{=} \frac{8\pi}{c^3}\nu^2 d\nu$$

$$\Rightarrow u(\nu)\,d\nu = h\nu\, n(\nu)\,d\nu = h\nu \frac{8\pi}{c^3}\nu^2 \frac{d\nu}{e^{h\nu/KT}-1}$$

$$\Rightarrow \boxed{\text{energy density } u(\nu)\,d\nu = \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/KT}-1}}$$

$$\boxed{u(\lambda)\,d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda KT}-1}}$$

$\nu = c/\lambda$

$d\nu = c/\lambda^2\, d\lambda$

EXERCISE 4: Calculate the total energy density of the photon field in thermodynamic equilibrium, and comment on the Stefan-Boltzmann's law.

$$u = \text{total energy density} := \int_0^\infty u(\nu)\,d\nu$$

$$u = \int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/KT}-1} = \frac{8\pi h}{c^3}\left(\frac{KT}{h}\right)^4 \underbrace{\int_0^\infty \frac{x^3 dx}{e^x-1}}_{\pi^4/15} = \frac{8\pi^5 K^4}{15 c^3 h^3} T^4 = a T^4$$

$x = h\nu/KT$

$\mapsto \nu = \frac{KT}{h}x$

$d\nu = \frac{KT}{h}dx$

w/ $a = 7.565 \times 10^{-15}\ \text{erg}\ \frac{1}{\text{cm}^3\ {}^\circ K^4}$

⇨ The temperature of a gas is defined by matching the observed distribution of particle states to the appropriate equilibrium distribution for the same type of particles.

If the observed distribution functions do not match the equilibrium distributions, the gas is not in thermodynamic equilibrium, & a concept of temperature must be employed w/ caution.
The stellar interior is nearly in the state of thermodynamic equilibrium.

Assigning a Temperature to the photosphere → match the energy spectrum of the photons leaving the surface of the star to those leaving a surface in thermodynamic equilibrium, ie: a BLACKBODY.

Power radiated per unit of area per unit of wavelength [erg/s/cm²/Å] $= \frac{c}{4} \cdot u(\lambda) = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} := I_\lambda$ (ie. F_λ) BLACK BODY

Total power radiated per unit of area [erg/s/cm²] $I = \int_0^\infty I_\lambda \, d\lambda = \frac{2\pi^5}{15} \frac{1}{c^2} \frac{k^4}{h^3} T^4 = \sigma T^4$

$x = hc/\lambda kT$

$\int_0^\infty \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$

w/ $\sigma = 5.67 \times 10^{-5} \frac{erg}{cm^2\, s\, {}^\circ K^4}$ Stefan-Boltzmann for a black body

monochromatic luminosity $L_\lambda = 4\pi R^2 \cdot I_\lambda$

$[L_\lambda] = erg/s/Å$

4 simple concepts of stellar temperature in common use for the photosphere:

- effective temperature : T_{eff}
- color temperature : T_{col}
- excitation temperature : T_{exc}
- ionization temperature : T_{ion}

These are different because the photospheres of stars are not in thermodynamic equilibrium.

$\boxed{T_{eff}}$ Defined as the temperature of a blackbody w/ the same radiated power per unit area I_λ

$\Rightarrow L = 4\pi R^2 \sigma T_{eff}^4$ (used to estimate stellar radii) $\sigma = 5.67\times10^{-5} \frac{erg}{cm^2 s K}$

EXERCISE 5: $r_\odot = 16$ arcmin from Earth.
I_λ (@ Earth's atmosphere) $= 1.388\times10^6$ erg/s/cm² (AKA the Solar constant)
? T_{eff} w/ only the given data.

$\boxed{T_{col}}$ It also matches the radiated photon power to a blackbody, but the matching is to the shape of the continuous spectrum I_λ rather than the integrated power

$$m_2 - m_1 = 2.5 \log\left(\frac{\ell_{\lambda_1}}{\ell_{\lambda_2}}\right) = 2.5 \log\left(\frac{I_{\lambda_1}}{I_{\lambda_2}}\right) + C$$

assuming no dust extinction

$$= \underbrace{A}_{\text{constant}} + \frac{1.56}{T_{col}}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + f(T_c)$$

function that varies slowly w/ T_c

$$\Rightarrow B-V = -0.586 + \frac{6850}{T_{col}} + f(T_{col})$$

[The hotter, the bluer $\Rightarrow$ smaller B-V]

EXERCISE 6: Calculate the position of the maximum of the blackbody spectrum in the form $\lambda_{max} T =$ const. For the Sun, $\lambda_{max} = 5000$ Å. Compute the Sun's T_{col} & compare it w/ T_{eff}.

SOLUTION: $\frac{du(\lambda)}{d\lambda} = 0 \Rightarrow \lambda_{max} T = 0.2014 \frac{hc}{k} = \underbrace{0.2898 \text{ cm K}}_{\text{Wien's constant}}$

$\downarrow$

Wien's displacement law

Fig 1-5

Differences between photosphere spectrum & black body:

- sizable discontinuities in the level of the continuous as the energy of the radiation crosses absorption thresholds.
  eg: drop @ $\lambda < \lambda^{limit}_{Balmer}$, since these photons are capable of ionizing that fraction of hydrogen atoms lying in the first excited state.
- composition of photosphere influences the shape of the continuous spectrum.
  eg: ionized metals have numerous resonance lines in the UV
  ↳ less UV radiation than a metal-deficient star of equal T.

$\boxed{T_{exc}}$ ⟷ Boltzmann equation (✱) → 16b

$\boxed{T_{ion}}$ obtained from the distribution of atoms in the ionized levels from the Saha's law.

By comparing the strength of absorption lines of two different states of ionization of the same atom, it is possible to determine the relative number densities of the two stages of ionization of that atom (eg. H & K lines of CaII w/ the line of neutral Ca (CaI) @ 4226 Å). The population ratio can be used to determine a stellar temperature.

SAHA's EQUATION

written in terms of electron density

$$\left(\frac{n_{r+1}}{n_r}\right) n_e = \frac{G_{r+1}}{G_r} g_e \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_r/kT}$$

partition functions ($g_e = 2$)

w/ n_{r+1} # density of the $(r+1)$-times-ionized species of an atom

n_r # density of the r-times-ionized species of the same atom

n_e # density of free e's

χ_r ionization energy of species r

written in terms of e's pressure

$$\Longleftrightarrow \left(\frac{n_{r+1}}{n_r}\right) P_e = \frac{G_{r+1}}{G_r} 2 \frac{(2\pi m_e)^{3/2}}{h^3} (kT)^{5/2} e^{-\chi_r/kT}$$

w/ $P_e = n_e kT$ (for a perfect non-degenerate gas of e's)

Boltzmann's equation

$$\frac{N_b}{N_a} = \frac{g_b}{g_a} e^{-(E_b - E_a)/KT}$$

different excitation states

$N_b, N_a :=$ # of atoms w/ energy E_b & E_a

$g_b, g_a :=$ statistical weights of the energy levels E_b & E_a

g_b is the # of states w/ energy E_b.

EXERCISE: Gas of HI (neutral H)

? At what T $N_{n=1} \equiv N_{n=2}$

$n=1$ ground state

$n=2$ 1st excited state

For Hydrogen, $\boxed{g_n = 2n^2}$

$E_n = -13.6\,eV \frac{1}{n^2}$

$$\Rightarrow 1 \equiv \frac{N_2}{N_1} = \frac{2(2)^2}{2(1)^2} e^{-[(-13.6\,eV/4) - (-13.6\,eV)]/KT}$$

$$E_2 = \frac{-13.6\,eV}{2^2}$$

$$\Rightarrow \frac{10.2\,eV}{KT} = \ln(4) \longrightarrow \boxed{T \simeq 8.54 \times 10^4\,K}$$

ie: high T are needed for a significant # of H atoms to have e^-s in the 1st excited state

$$\lg \frac{n_{r+1}}{n_r} = \lg \frac{G_{r+1}}{G_r} + 15.6839 + \frac{3}{2}\lg T - \frac{5039.78}{T}\chi_r - \lg n_e \qquad (17)$$

w/ $[\chi_r] = eV$ $\quad [n_e] = cm^{-3}$

$[T] = {}^\circ K$ $\quad [p_e] = ba$

$$\lg \frac{n_{r+1}}{n_r} = \lg \frac{G_{r+1}}{G_r} - 0.176 + 2.5 \lg T - \frac{5039.78}{T}\chi_r - \lg P_e$$

NOTE: In environments where deviations from local thermodynamic equilibrium are large (e.g., solar corona, planetary nebulas, etc.), the Saha eq. is not applicable.

NOTE: From the strengths of the lines, I can derive T_{ion} & P_e (note that in main-sequence stars, $P_e \simeq$ const). Vice versa, from T and P_e, I can derive N_{r+1}/N_r, ie: the relative abundance of elements (see Fig 8.11 in your book)

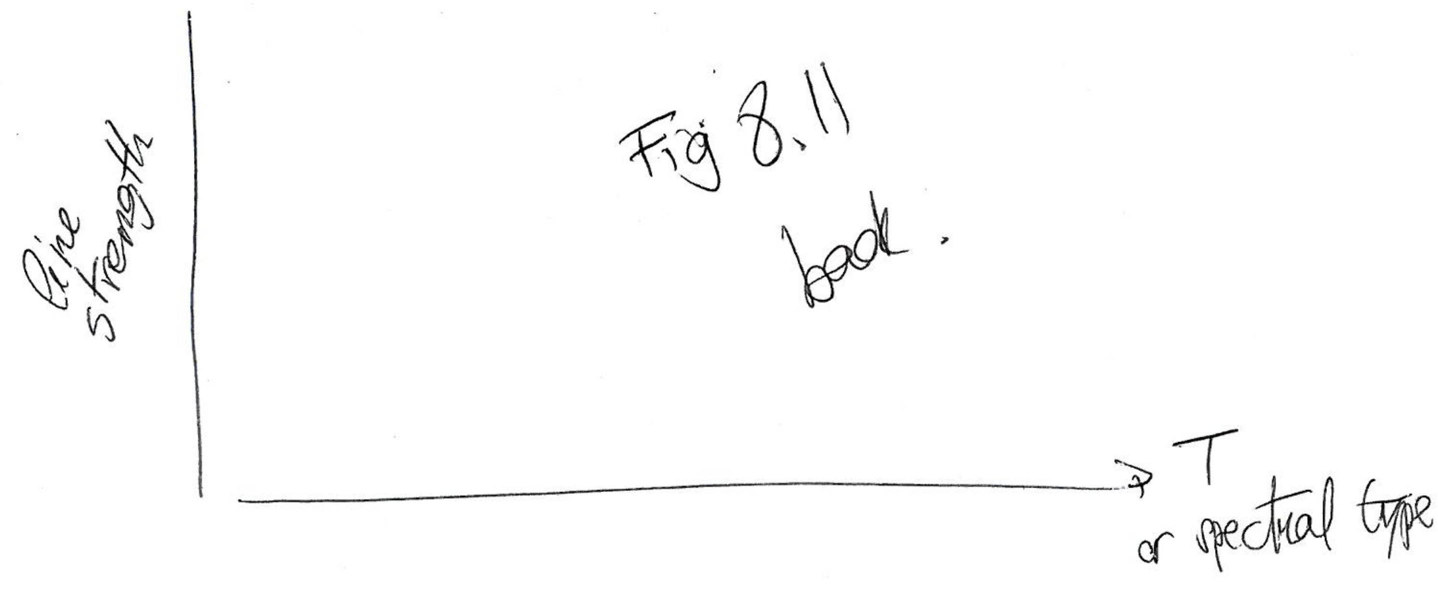
☐ IF hydrogen gas is in thermodynamic equilibrium

$$\frac{N_m(H)}{N(H^+)} = \frac{n_e h^3 n^2}{(2\pi m_e kT)^{3/2}} e^{\chi_H / m^2 kT}$$

level #

# of H atoms in energy level m

# of ionized hydrogen ions



EXERCISE: Photosphere of pure Hydrogen; $P_e = 20\ N/m^2$ constant (17b)

Saha's eq. $\longrightarrow$ fraction of ionized atoms as a function of T.

$$\frac{N_{II}}{N_{Total}} = \frac{N_{II}}{N_I + N_{II}} \equiv \frac{N_{II}/N_I}{1 + N_{II}/N_I}$$

$H_{II} \equiv p \rightarrow$ no degeneracy $\Rightarrow G_{II} = 1$,

Energy of 1st excited state is $E_2 - E_1 = 10{,}2\ eV \gg KT$ for $T \in [5, 25] \times 10^3\ K$

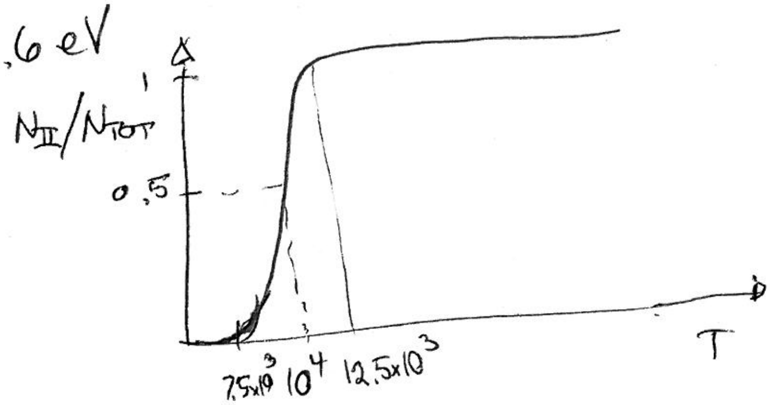
$\Rightarrow e^{-(E_2-E_1)/KT} \ll 1$ in Boltzmann's eq.

$\Rightarrow$ ~ all H_I is in ground state

$\Rightarrow G_I \simeq g_1 = 2(1)^2 = 2$

Saha's eq: $$\frac{N_{II}}{N_I} P_e = \cancel{\frac{1}{2}} \cdot \cancel{2} \frac{(2\pi m_e KT)^{3/2}}{h^3} e^{-13.6\ eV/KT} \cdot KT$$

$\chi_I = 13{,}6\ eV$



$\Rightarrow \dfrac{N_{II}}{N_{TOT}} = \ldots$

50% ~ T = 9600 K

$\Rightarrow$ Ionization of H w/in a T interval of ~3000 K.

Strength of Balmer line $\propto N_2/N_{TOT} = \left(\dfrac{N_2}{N_1+N_2}\right)\left(\dfrac{N_1}{N_{tot}}\right) = \dfrac{N_2/N_1}{1+N_2/N_1} \dfrac{1}{1+\dfrac{N_{II}}{N_I}}$

$N_1 + N_2 = N_{Tot} \simeq N_1$

Boltzmann's eq.

Saha's eq.

Ca in the Sun: $\chi_{I,Ca} = 6.11$ eV

$Z_I = 1.32$, $Z_{II} = 2.30$

$\longrightarrow \left.\frac{N_{II}}{N_I}\right|_{Ca} = 918$ for $T = 5777K$, $P_e = 1.5\ N/m^2$

500,000 H every Ca

$\longrightarrow$ practically all Ca is in the form of Ca II.

Ca K ($\lambda = 3933 Å$) Ca II 1st excited state $E_2 - E_1 = 3.12$ eV

$g_1 = 2$, $g_2 = 4$

$\Rightarrow \left.\frac{N_2}{N_1}\right|_{Ca II} = \frac{1}{264}$ ie; out of every 265 Ca II ions, all but one are in the ground state & are capable of producing the Ca II K line.

Boltzmann

$\Rightarrow$ ~ all Ca atoms in the Sun are singly ionized & in the ground state, hence available for forming H & K lines.

$$\left.\frac{N_1}{N_{TOT}}\right|_{Ca II} \simeq \left.\frac{N_1}{N_1+N_2}\right|_{Ca II} \left.\frac{N_{II}}{N_{Tot}}\right|_{Ca} =$$

$$= \frac{1}{1 + \left.N_2/N_1\right|_{Ca II}} \cdot \frac{\left.N_{II}/N_I\right|_{Ca}}{1 + \left.N_{II}/N_I\right|_{Ca}} = 0.995$$

There are ~ 500K H atoms for every Ca, but only a tiny fraction (5×10^{-9}) are un-ionized in the 1st excited state, capable of producing a Balmer line.

The strength of the H & K lines is NOT due to greater abundance of Ca, but reflects the sensitive T dependence of the atomic states of excitation & ionization.

Maxwell-Boltzmann Velocity Distribution

$$\frac{dn(v)}{dv} = n\left(\frac{m}{2\pi KT}\right)^{3/2} e^{-mv^2/2KT}\, 4\pi v^2$$

# of gas particles per unit volume w/ speeds between v & $v+dv$

n = total # density $= \int_0^\infty n(v)\,dv$

Peak of the distribution (most probable speed): $V_{peak} = \sqrt{\frac{2KT}{m}}$

$$\frac{d}{dv}\left(\frac{dn}{dv}\right) = 0$$

Average speed $(\sqrt{\bar{v}^2})$ $\quad v_{rms} = \sqrt{\frac{3KT}{m}}$, w/ $v_{rms} > V_{peak}$ due to the high-speed exponential tail.

→ Kinetic temperature

$$V_{ave} = \sqrt{\frac{8KT}{\pi m}}$$

$$\boxed{v_{peak} < v_{ave} < v_{rms}}$$

some useful integrals

$$v_{ave} := \frac{\int_0^\infty v\, dn(v)}{N}$$

$$\overline{v^2} = \frac{1}{N}\int_0^\infty v^2\, dn(v)$$

$$\int_{-\infty}^{+\infty} \begin{cases} 2I_n & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

$$I_n = \int_0^\infty x^n e^{-ax^2}\,dx$$

n	I_n	n	I_n
0	$\frac{1}{2}\sqrt{\pi/a}$	1	$1/2a$
2	$\frac{1}{4}\sqrt{\pi/a^3}$	3	$1/2a^2$
4	$\frac{3}{8}\sqrt{\pi/a^5}$	5	$1/a^3$
6	$\frac{15}{16}\sqrt{\pi/a^7}$	7	$3/a^4$

Spectral Types

Classification (empirical) in which stars are sorted into 7 spectral types, according to their surface temperature & appearance of characteristic absorption lines (in the optical)

[O] : $T > 25 \times 10^3\,°K$; prominent lines of ionized He

[B] : $T \in (11 \times 10^3, 25 \times 10^3)\,K$; lines of H & neutral He are conspicuous @ B∅ ; ionized oxygen & carbon became strong at B3. Neutral He lines strongest at B5. Hydrogen lines became progressively stronger (Balmer lines in absorption)

[A] : $T \in (7.5 \times 10^3, 11 \times 10^3)\,K$; in A∅, hydrogen & ionized magnesium lines are strongest, while He & ionized oxygen lines have disappeared; H lines weaken, whereas ionized metals (Fe, Ti, Ca, etc.) strengthen.

[F] : $T \in (6, 7.5) \times 10^3\,K$; F∅ rich of ionized metal lines, strongest being H & K lines of Ca II (singly-ionized Ca); Metallic lines (eg, Fe) strengthen.

[G] : $T \in (5, 6) \times 10^3\,K$; lines of neutral metals became strong, while hydrogen lines continue to weaken; lines of ionized Ca are very strong ; molecular bands of CN & CH appear.
Sun = G2

[K] : $T \in (3.5, 5) \times 10^3\,K$; molecular bands & lines of neutral metals are much stronger ; at K5, lines of TiO are weakly visible

[M] : $T \in (2.2, 3.5) \times 10^3\,K$; appearance of complex molecular bands, of which TiO bands are strongest. Oxide bands, of which TiO bands are strongest.

Additional classes :

[S] : low-T parallel of [M]. Most prominent feature is ZrO bands; elements Zr, Y, Ba, La, Sr give strong atomic lines & oxide bands.

[R & N] (or [C]) : parallel in temperature to [K] & [M]. Spectrum characterized not by oxide bands, but molecular carbone bands, eg CN, C_2, CH.

[W] : extremely high-T type O objects, Wolf-Rayet stars, w/ bright, broad emission lines of ionized He & highly ionized C, O, N. These stars emit gas rapidly in space.

→ show Fig 1-11 (pag 38).

Hydrogen spectrum

$$\frac{1}{\lambda} \equiv \sigma = R Z^2 \left(\frac{1}{n_o^2} - \frac{1}{n^2} \right)$$

w/ Z atomic number (# of protons)
R Rydberg constant
n_o principal quantic number

$$R = \frac{m_e e^4}{8 \epsilon_0^2 h^3 c}$$

$Z = 1$ (H) → $R_H = 109677.583 \text{ cm}^{-1}$

$n \to \infty$ ⇒ limit of the series & $\sigma \to \sigma_o = \frac{R Z^2}{n_o^2}$

Lyman series ($n_o = 1$) : $n = 2 \to L_\alpha = 1215.67$ Å $(2 \leftrightarrow 1)$ $L_\beta = 1025.72$ Å $(3 \leftrightarrow 1)$
(all in the UV) $\sigma_o \leftrightarrow \lambda_o = 911.8$ Å (Lyman break)

Balmer series ($n_o = 2$) : $n = 3 \to H_\alpha = 6562.81$ Å $(3 \leftrightarrow 2)$
(visible & near UV) $n = \infty \to \lambda_o = 3645.95$ Å (Balmer break)

$H_\beta = 4861.34$ $(4 \leftrightarrow 2)$
$H_\gamma = 4340.48$ $(5 \leftrightarrow 2)$
$H_\delta = 4101.75$
$H_\varepsilon = 3970.07$
$H_8 = 3889.05$

Paschen series ($n_o = 3$) : $n = 4 \to P_\alpha = 1.8751\ \mu m$
(NIR) $n = \infty \to \lambda_o = 8204$ Å

Brackett series ($n_o = 4$) : $n = 5 \to B_\alpha = 4.05\ \mu m$
(NIR in K-band) $B_\beta = 2.63\ \mu m$

$Pa\beta = 1.28181$
$Pa\gamma = 1.09381$

Helium : $Z = 2$ $R_{He} = 109722.26\ cm^{-1}$ (He II)

$n_0 = 2$ 1 ($\neq$ UV)

Fowler series ($n_0 = 3$) : $F_\alpha = 4685.7$ Å

$F_\beta = 3203.1$ Å

Pickering series ($n_0 = 4$) : similar to Balmer lines

Metals: anything heavier than He.

C IV important in the UV for studies of cranosphere

Si II 4130.9 Å
4128.1 Å

Na 5895.9 Å
5890. Å

Ca I 4226.7 Å

Ca II H & K 3968.5 Å (blending w/ Hϵ)
3933.7

Mg II 4481.3 Å
5183.6
5172.7
5167.3

$\left.\begin{array}{l} 5183.6 \\ 5172.7 \\ 5167.3 \end{array}\right\}$ strongest lines in the Sun

Mg II h K 2795 Å
2800

MASS

- can only be measured when they occur in a binary pair & the orbital motion of the pair is measured.

$$\Rightarrow \frac{M_1 + M_2}{M_\odot + M_{Earth}} = \frac{A^3}{P^2}$$ w/ A semi-major axis between the two stars $[A] = AU = 1{,}496 \times 10^{13}$ cm

P period of the binary system $[P] = yr$

If the center of mass of the pair can be found $\longrightarrow M_1/M_2$.

- $M_\odot = (1{,}989 \pm 0{,}002) \times 10^{33}$ g Sun.

$\boxed{\cdot}$ $L = \text{const} \cdot M^\beta$, $\beta \in (3.5, 4)$ for main-sequence stars

This law is violated by white dwarfs, giants, ...

RADIUS

- This quantity, extremely important for theory of stellar evolution, have not generally been measurable (only recently w/ interferometry).
- $L = 4\pi R^2 \sigma T_e^4 \longrightarrow R$ once T_e & L are measured from independent techniques.

Limitations due to the fact that the photosphere is not a perfect black body & by convection zones near the surface.

- $R_\odot = (6.9598 \pm 0{,}0007) \times 10^{10}$ cm.

MEASURABLES: L, $T_{surface}$, R, M

spectral features $\longrightarrow$ composition of stellar surfaces

$\Downarrow$

Theory of stellar evolution

ENERGETICS:

- Temperature, pressure, & density in the interior of a star are related by the equation of state of the gas, ie: $P = P(T, \rho)$.
- The rate at which energy is transported from the interior to the surface to be radiated depends on the temperature gradient of the stellar structure.
- T_{core} must have the proper value for nuclear reactions to proceed at the required rate to counterbalance the luminosity. Moreover nuclear reaction rates are very strongly temperature-dependent $\longrightarrow$ severe restrictions on T of stellar interiors.
- Interior heat is removed in 4 ways:
  1. conduction, depending on collisions between particles, during which thermal energy is transferred via elastic collisions (slow process of energy transport, extremely inefficient in a gaseous state, except for degenerate e^-s gases)
  2. radiation (radiative transfer); thermal emission is proportional to T^4 $\longrightarrow$ a hot spot radiates energy faster than it receives from cooler surroundings. If opacity of gas not too large $\longrightarrow$ energy transported very efficiently by photons
  3. Convection: a mass of gas in a hot region moves bodily to a cooler region, redistributing it's excess thermal energy. Relatively efficient.
  4. Neutrinos: escape w/o interaction

SOURCE OF ENERGY

1. WORK DONE by GRAVITATION CONTRACTION

Total gravitational binding energy of Sun

$$\Omega = -\frac{GM^2}{R} = -\frac{(7\times10^{-8}\ \frac{cm^3}{g\,s^2})(2\times10^{33}\,g)^2}{7\times10^{10}\,cm} = -4\times10^{48}\,erg$$

If energy radiated at present luminosity $t_\odot L_\odot = 4\times10^{48}\,erg$

w/ $t_\odot$ age of Sun, $L_\odot = 3.9\times10^{33}\,erg/s$ $\Rightarrow$ $t_\odot \approx 10^{15}\,s = \underline{3\times10^7\,yr}$ $\ll t_{earth} = 4.6\times10^9\,yr$

2. Conversion of Rest mass into Kinetic energy according to $E = mc^2$ (24)

$$M_\odot c^2 = (2\times10^{33} g)(3\times10^{10} cm/s)^2 = 2\times10^{54} erg/s$$

Stars are predominantly H, about 90% of all nuclei. Most abundant source of energy in stellar interiors is the fusion of 4H into He.

$$(4M_H - M_{He})c^2 = 26.73 \text{ MeV}$$

$$4M_H \simeq 4.931 \text{ MeV}$$

$\Rightarrow$ Fraction converted $= \dfrac{26.73 \text{ MeV}}{4,931 \text{ MeV}} = 0,007 = \boxed{7‰}$

ie: the transmutation of H into He liberate 0.7% of the rest mass of the system in the form of energy

$$\Rightarrow \frac{2\times10^{54} \frac{erg}{s} \cdot 0,007}{4\times10^{33} \frac{erg}{s}} = 3\times10^{18} s \simeq 100 \text{ Gyr lifetime}$$

EXERCISE: 1 amu = $1,66053886\times10^{-27}$ g
$M_H = 1.007825$ amu $M_{He} = 4,002603$ amu
? Energy release by nuclear binding when 4Hs combine to form He.
? How much energy is liberated when 1g of H is converted to 1 g of He?
ANSWER = $6,4\times10^{18}$ erg/g

Problem: Coulomb repulsion between protons
The work done in bringing two charges to a separation r is the potential energy V(r)

$$V = \frac{9,92}{r} = \frac{1,44\, Z_1 Z_2}{r(fm)} \text{ MeV} \qquad 1 fm = 10^{-13} cm.$$

For nuclear reactions to occur, two protons must come to a separation of ~2 fm $\longrightarrow$ V = 0.7 MeV = 700 keV, ie: two classical particles w/ a relative kinetic energy less than 700 keV could not come close enough for interaction.
The average thermal energy in a gas in thermal equilibrium is

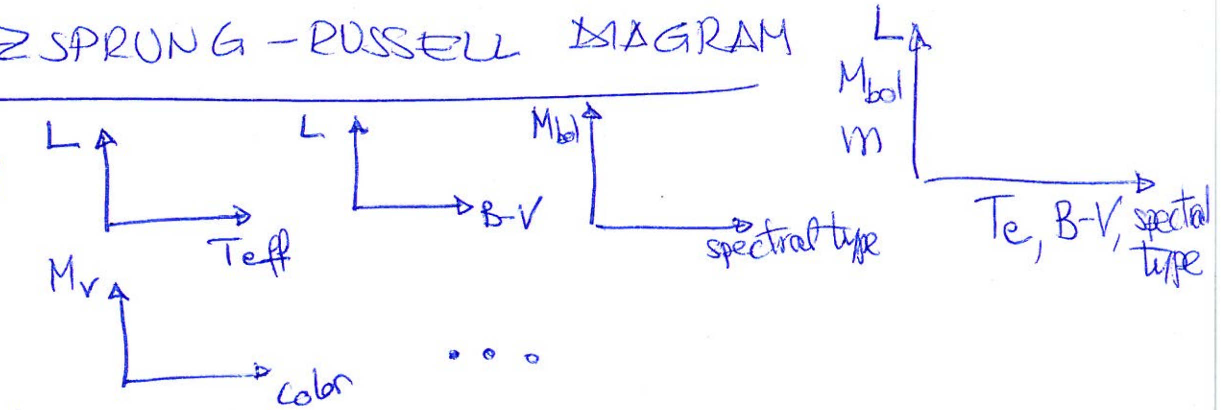
$$KT = 0,862\times10^{-7} T \text{ KeV} \simeq 1 \text{ KeV} \ll 700 \text{ KeV} !!$$

$T = 10^7$ K @ Sun's center

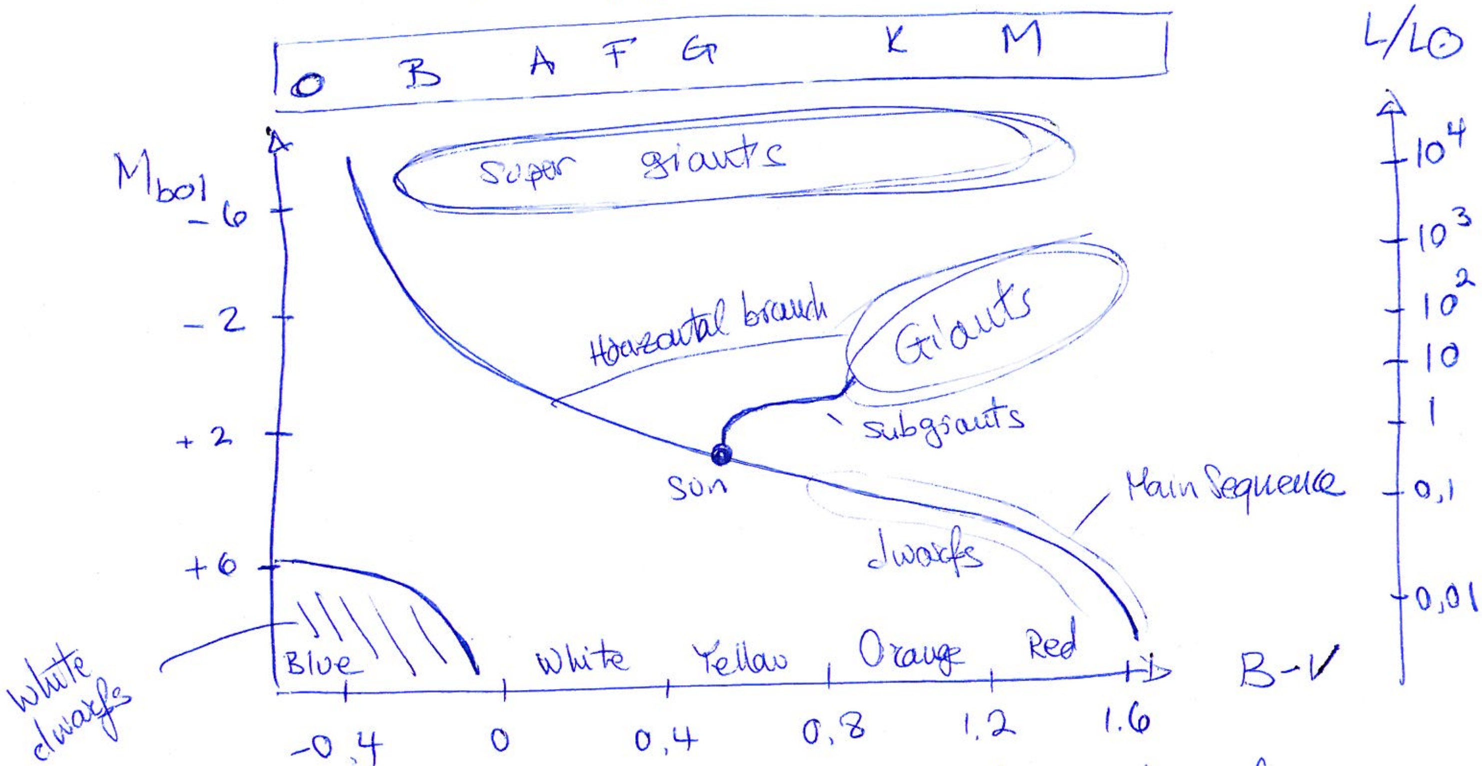
However particles may, w/ low probability, penetrate through the barrier, quantum-mechanical effect of TUNNELING
→ fusion reactions can proceed in the interiors of stars @ a sufficient rate to account for their luminosity.

THE HERTZSPRUNG-RUSSELL DIAGRAM

HR diagrams:



For distant clusters



Main Sequence stars: brightest objects are those w/ highest surface temperatures & bluer colors. 80-90% of observed stars. Burning H in stellar interiors as their source of energy.

EXERCISE: If an O5 star had $T_{surf} = 35000\,°K$ black body, estimate the peak in the continuous spectrum ($\lambda = 830\,\text{Å}$). What fraction of the energy radiated by an O5 star is visible? (1.4%)

EXERCISE: Calculate the radii of B0, A5 & M0 main-sequence stars. Compare to $R_\odot$

	$\log L/L_\odot$	T_{eff}
B0	$= 4.56$	$= 21000°K$
A5	1.20	$= 8100°K$
M0	-1.15	$= 3300°K$

Red giants: redness is due to large radii; only a few percents

Subgiants: stars whose envelopes are expanding while their He cores contract to a point where He begins to produce energy by nuclear reactions.

Horizontal branch: various phases of He burning

Supergiants: advanced stages of stellar evolution & approaching the end of their energy-generating lifetime.

White dwarfs: much smaller than Sun but w/ comparable mass, i.e. very high densities, high T_{surf}, hence blue or white. Up to 10% or so of all stars. They are the stellar graveyard, consisting of degenerate matter (e^-s filled all available cells in momentum space) → large internal pressure supporting the structure. No internal energy source left, w/ residual supply of thermal energy slowly radiated into space.

$R_{wd} \sim \frac{1}{100} R_\odot$

EXERCISE: Calculate R_{wd} w/ $L = 10^{-2} L_\odot$ & $T_e = 10^4$ K.

NOTE: H-R diagram of clusters of stars
1. all @ the same distance
2. all w/ the same age (modulo 10^8 yrs)

- STAR CLUSTER: group of stars w/ a much stronger gravitational attraction to each other than to general field stars. Number of stars varies from 10^5 to loose associations of a few stars.

Richest clusters are massive, spherical ones w/ 10^5 stars: $\boxed{\text{globular clusters.}}$

Their distances from apparent magnitudes of RR Lyrae variable stars

Typical diameter of the high-star-density regions is tens of pc, w/ $\rho_{central}$ as high as 10^3 stars/pc^3!

EXERCISE: If $M_V^{RRLy} = 0$ & $m = 15$, what is the distance to the cluster?

$\boxed{\text{open clusters}}$: more irregular groupings of a few to a few hundred stars. Found only in the disks of galaxies.

ASSOCIATION: special type of open cluster w/ most luminous main sequence O & B stars (& WR stars) over dimension of ~100 pc.

Pop I: relative young stars w/ blue giants as most luminous members

Pop II: old stars w/ red giants " " " "

H-R diagram of Pleiades vs Globular cluster M3:

evolution of the stars from the upper tip of the remaining main sequence into the giant region $\longrightarrow$ old age of globular clusters

NOTE: $L \propto M^\beta$, $\beta \in (3.5, 4)$

lifetime of star $\sim \frac{M_H}{L} \cdot \text{const} \propto \frac{M}{L} \propto \frac{M}{M^\beta}$

$\Rightarrow \boxed{\text{Lifetime} \propto M^{1-\beta}}$

$\Rightarrow t \simeq \underbrace{10\ Gyr}_{t_\odot} \left(\frac{M}{M_\odot}\right)^{-3}$

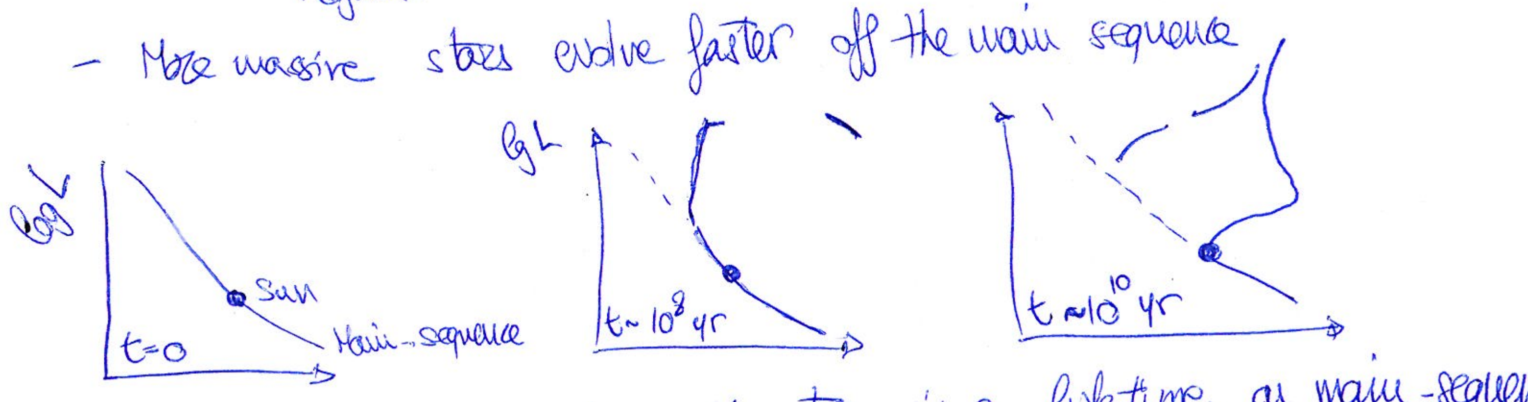
see below better...

STELLAR EVOLUTION

By careful interplay of auxiliary observations w/ theoretical arguments, stars of the snapshot could be ordered w/ respect of age.

1. Theory of polytropic gas spheres → equation of state of the gas combined w/ the condition of hydrostatic equilibrium to yield the properties of dynamically stable gas sphere. (Emden, 1907)
2. Energy transported from the center to the surface by radiative transfer → theory relating energy flux to temperature gradient & opacity (Eddington 1930)
3. White dwarfs → degenerate matter plays a key role in many stages of stellar evolution (Chandrasekhar 1939)
4. Nuclear reactions (Hoyle, Schwarzschild, Henyey) 1958

- Stars form from gas in the ISM, composed predominantly of H
- They contract until T_c high enough for H thermonuclear reactions to begin. At this point they radiate energy @ a rate equal to that liberated by the nuclear reactions. They remain static as long as there is H fuel in the core.
- when innermost 10-20% of H in core has been exhausted, the outer regions expand & the inner core contracts.
  → same luminosity but lower Tsurf, hence redder color.
  → the star evolves off the main sequence & into the giant region.
- More massive stars evolve faster off the main sequence



- total energy radiated from the star in a lifetime on main-sequence

$$E = \underbrace{f}_{\text{fraction of mass before H burning stops in the core}} \underbrace{X_H}_{\text{fraction of H}} \underbrace{M}_{\text{Mass of star}} \underbrace{(6.4\times10^{18}\ \text{erg/g})}_{\text{energy released by } 4H \rightarrow He} = 1.3\times10^{52} f X_H \frac{M}{M_\odot}\ \text{erg}$$

- life on main-sequence $t_E = \frac{E}{L} = 1.1 \times 10^{11} f X_H \frac{M/M_\odot}{L/L_\odot}$ yr

NOTE: $f \sim 15\%$

$$\longrightarrow t_E = 12 \times 10^9 X_H \frac{M/M_\odot}{L/L_\odot} \text{ yr}$$

But $M/M_\odot \approx (L/L_\odot)^{1/4}$

Using $X_H = 0.6 \Rightarrow t_E \simeq 12 \left(\frac{L}{L_\odot}\right)^{-3/4} \times 10^9$ yr.

EXERCISE: Estimate t_E for a star w/ $M_{vis} = 3$ & $M_{vis} = -2$.

$\Rightarrow$ The most luminous stars in a stellar cluster still on the main sequence yield the age of the cluster (in details, much more complicate calculations).

Fig 1-21 comparison of different clusters $\longleftrightarrow$ age.

$\Rightarrow$ The crucial point for determination of the age of the cluster lies in determining the absolute magnitude of the turn off point, or the surface temperature of the turnoff point.

NOTE: Because the central composition changes during the life on the main sequence, L & R both increase $\rightarrow$ nearly vertical small rise.
$\Rightarrow$ the main sequence rises slightly w/ age
This is due to the collapse of the hydrogen-depleted core..

- When H burning stops, the central region becomes unable to generate sufficient pressure to support the overlying layers. As it collapses, T rises because of gravitational work. The increase of $T_{central}$ causes outer layers to expand in order for the temperature gradient not be too great, hence reddening of the surface.

- . . . white dwarfs details of mass loss not well known.
supernova

synthesis of most heavy elements . . .

NUCLEOSYNTHESIS

- Nucleosynthesis occurs in the natural evolution of stars.
- The initial H & He are fused into heavier nuclei, dispersed into
- the ISM in the terminal phases of the stellar lifetime.
- During the first 3 min, H (75%), He (25%) & traces of D^2, He^3, Li, Be, B were produced (primordial abundances)
- Show Fig 1-22, abundance of elements
- C^{12}, O^{16}, Ne^{20} most abundant elements after H & He (He fusion)
- Fe^{56} : nuclear binding energy per nucleon has a maximum at Fe^{56}, ie: successive nuclear fusion reactions cease to liberate energy when all light nuclei have been fused into Fe^{56}. Further fusion requires energy $\Rightarrow$ termination of energy-generating stages of nuclear fusion.
- Very heavy nuclei can be formed efficiently by the capture of free neutrons liberated as a by-product of reactions between light charged particles.