① The luminosity of a static star is determined by two independent conditions:

1) rate of energy flow, ie: the luminosity L, is determined by the temperature gradient dT/dr & the details of energy transport

2) the luminosity of a static star = rate at which energy is liberated by nuclear reactions

$\Rightarrow$ details of energy generation & transport in the stellar interior

① Transport of energy is caused by dT/dr
- radiative transfer $\propto dT/dr$
- convection
- conduction $\propto dT/dr$
- neutrino

when dT/dr very large, the gas develops convective instabilities

ENERGY BALANCE (energy conservation)

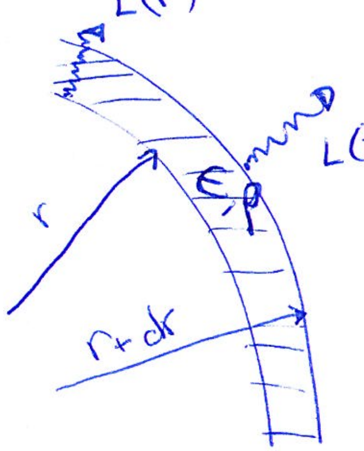
- most stars are static, ie: $T(r)$, $\rho(r)$, $P(r)$ hardly change w/ time $\Rightarrow$ power radiated from the surface must be supplied at the same rate in the stellar interior

$$\epsilon = \varepsilon(\rho, T, \{X_Z\})$$

power liberated per gram of stellar matter by nuclear reactions. $[\varepsilon] = erg/s/g$

$\{X_Z\}$: composition

(this does not include what is carried away by ν_s) $\epsilon = E_{total} - E_{\nu}$



$\Rightarrow \rho\epsilon$ = rate at which energy is liberated per cubic centimeter of stellar matter

$[\rho\epsilon] = erg/s/cm^3$

For a static stellar structure

$$L = \int_V \epsilon\rho\, dV = \int_0^R 4\pi r^2 dr\, \epsilon\rho$$ total luminosity of the star.

NOTE: No star is exactly static. Nuclear reactions change $\{X_Z\}$, leading to very slow changes $\Rightarrow$ sequence of static structures.

2

$L(r+dr) - L(r) = \epsilon(r)\rho(r)\, 4\pi r^2 dr$ balance of energy in the shell at radius r.

$\xLeftrightarrow[\text{equiv.}]{dL}$ $\boxed{\dfrac{dL(r)}{dr} = \epsilon(r)\rho(r)\, 4\pi r^2}$ $\boxed{4}$ major relationship to be satisfied by a static stellar structure.

Since $4\pi r^2 \rho(r) dr = dM(r)$

$\Rightarrow$ $\boxed{\begin{array}{l} \dfrac{dL(r)}{dM(r)} = \epsilon(r) \\ M(r) = \int_0^r 4\pi r^2 \rho\, dr \end{array}}$ $\boxed{4}$ expressed as a mass gradient instead of radial gradient.

This equality does not have to be maintained all the time. The star would contract & the virial theorem ensures that $\frac{1}{2}\Omega$ goes into kinetic energy & other half is radiated away, until settled into a new configuration. Re-configuration is a slow process.

When star is slowly contracting: $\dfrac{dQ}{dt} = \dfrac{dU}{dt} + P\dfrac{dV}{dt}$ (1st law of thermodynamics)

$\frac{dQ}{dt}$: amount of heat to be added per gram per second to stellar matter

$\frac{dU}{dt}$: rate of increase in internal energy

$P\frac{dV}{dt}$: rate at which the mass of gas does expansion work here $V = \frac{1}{\rho}$ is the specific volume

$\dfrac{dQ}{dt} = T\dfrac{dS}{dt}$ (2nd law of thermod.)

in the spherical shell

$\dfrac{dQ}{dt} = \epsilon(r) - \dfrac{L(r+dr) - L(r)}{\rho 4\pi r^2 dr}$ $\xLeftrightarrow[\text{eq.}]{}$ $\boxed{\dfrac{dL(r)}{dM(r)} = \epsilon - T\dfrac{dS}{dt}}$

4B important equation in the computation of stellar models during evolutionary change.

<u>EX:</u> For a gas, monoatomic w/ negligible radiation pressure

$S = \dfrac{K}{\mu m_H} \ln \dfrac{T^{3/2}}{\rho} + \text{const} =$ $\llcorner P = \dfrac{\rho K T}{\mu m_H}$

$= \dfrac{3K}{2\mu m_H} \ln \dfrac{P}{\rho^{5/3}} + \text{const}$

③

$$\Rightarrow T\frac{dS}{dt} = T\frac{3}{2}\frac{k}{\mu m_H}\frac{\rho^{5/3}}{P}\frac{d}{dt}\frac{P}{\rho^{5/3}} \quad \llcorner P = \frac{\rho k T}{\mu m_H}$$

$$= \frac{3}{2}\rho^{2/3}\frac{d}{dt}\frac{P}{\rho^{5/3}}$$

$$\Rightarrow \boxed{\frac{dL}{dM} = \epsilon - \frac{3}{2}\rho^{2/3}\frac{d}{dt}\frac{P}{\rho^{5/3}}}$$ gas monatomic w/ negligible P_{rad}

$$\frac{dL}{dM} = \epsilon - \frac{3}{2}\frac{k}{\mu m_H}T\frac{d}{dt}\ln\frac{e^{8y/3}\,y^{2/3}}{T}, \quad w/\ y = 1-\beta/\rho$$

when P_{rad} is significant but the gas is non-degenerate.

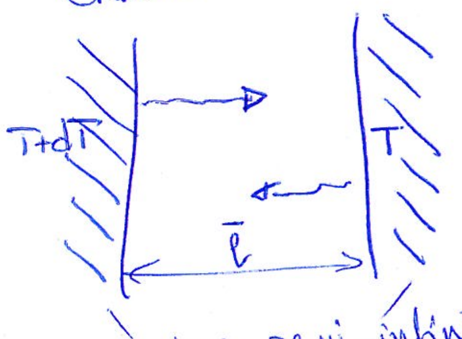
NOTE: Eq. 4B does not determine the luminosity of a star. The actual rate of flow of energy is determined by the mechanism of energy transport, which is determined by the dT/dr of the star.

RADIATIVE TRANSFER

The bulk of energy transport in stars occurs by the mechanisms of radiative transfer & convection.

NOTE: photons in the stars are able to travel only $\lesssim 1$ cm before interacting with matter $\Rightarrow$ it is the existence of a temperature gradient over distances of the order of ~1 cm that allows radiative transfer to occur.

Since $I \propto T^4$ (emission of blackbody from matter) $\Rightarrow$ we expect the rate of energy transport to be $\propto (dT/dr)^4$. The effectiveness of the transport depends on how opaque the gas is to the characteristic photons.



plane, semi-infinite blackbodies (a blackbody absorbs all radiation)

Net difference in the exiting flux over the entering flux per unit area is

$$H = \sigma\left[(T+dT)^4 - T^4\right] \approx 4\sigma T^3 dT$$

Heat flow in a vacuum between two parallel black bodies of temperature difference dT.

$$dT = \underbrace{\left(\frac{dT}{dx}\right)}_{\text{temperature gradient}} \bar{l}$$

$\bar{l}$ characteristic distance photons travel before absorption (average mean free path of photons)

$$\Rightarrow \quad H = \underbrace{-}_{\text{the heat flux is opposite to the temperature gradient}} 4\bar{l}\sigma T^3 \frac{dT}{dx}$$

Energy flux per unit area

The reduction in intensity of a beam of photons as it passes through matter of density ρ is

$$\frac{dI}{dx} = -\bar{\kappa}\rho I$$

w/ $\bar{\kappa}$ mass absorption coefficient, $\bar{\kappa} = \bar{\kappa}(\nu)$

$$\Rightarrow I(x) \propto e^{-\bar{\kappa}\rho x}$$

NOTE: $(\bar{K}\rho)^{-1}$ is the approximate distance ℓ that photons travel before absorption, ie: $\bar{\ell} = \dfrac{1}{\bar{K}\rho}$

$$\Rightarrow \boxed{H \simeq -\frac{ac}{\bar{K}\rho} T^3 \frac{dT}{dx}}$$

, ie: in the presence of a temperature gradient in a gas approximately in thermal equilibrium, the flux energy is proportional to the gradient of T^4 & to $(\bar{K}\rho)^{-1}$.

$$\sigma = \frac{ca}{4}$$

$$\bar{\ell} = \frac{1}{\bar{K}\rho}$$

A lengthy but correct calculation returns (see additional material)

$$\boxed{H = -\frac{4ac}{3K\rho} T^3 \frac{dT}{dr}}$$

w/

$$\boxed{\frac{1}{K} := \frac{\displaystyle\int_0^\infty \frac{1}{K_{\nu a}[1-e^{-h\nu/KT}] + K_{\nu s}} \frac{dB_\nu}{dT} d\nu}{\displaystyle\int_0^\infty \frac{dB_\nu}{dT} d\nu}}$$

Rosseland mean opacity K.

$K_{\nu a}$ = mass absorption coefficient of frequency ν associated to true absorption (when the energy of the photon interacting with the matter is converted into some other form of energy)

$K_{\nu s}$ = mass absorption coefficient of frequency ν associated to scattering absorption (the process by which the direction of the photon is changed).

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/KT} - 1} \qquad B_\lambda(T) = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{e^{hc/\lambda KT} - 1}$$

is the power radiated per unit of area per unit of wavelength by the blackbody of temperature T $\left(B_\nu(T) = \frac{c}{4} u(\nu)\right)$. Sometimes called SOURCE FUNCTION.

NOTE: The Rosseland mean opacity is an average over frequency, i.e.:

$$\frac{1}{\kappa} = \left\langle \frac{1}{\kappa_{\nu a}\left[1 - e^{-h\nu/KT}\right] + \kappa_{\nu s}} \right\rangle$$

6

w/ the average taken with respect to the weighting factor

$$\frac{dB_\nu}{dT} = \frac{2h^2\nu^4}{c^2KT}\,\frac{e^{h\nu/KT}}{\left(e^{h\nu/KT}-1\right)^2}$$

- The sources of radiative opacity are essentially additive except for the fact that the true absorption coefficients are reduced by the factor $1-e^{-h\nu/KT}$ to correct for induced emission.
- $\frac{dB_\nu}{dT}$: the photons frequencies most important for radiative transfer are those for which the difference in the product (photon number density · photon energy) between two points of slightly different temperature is maximal.

The net outflow of energy through a shell of radius r

$$L(r) = 4\pi r^2 H$$

$$\boxed{L(r) = -4\pi r^2 \frac{4ac}{3\kappa\rho} T^3 \frac{dT}{dr}}$$

basic eq. of stellar structure (when radiative transfer is the dominant mode of energy transport).

eq. $\Rightarrow \frac{dT}{dr} = -\frac{3\kappa\rho L(r)}{4\pi r^2 4acT^3}$ 5

Since $P_r = \frac{1}{3}aT^4 \rightarrow \frac{dP_r}{dr} = \frac{4}{3}aT^3\frac{dT}{dr}$

$$\Rightarrow \frac{dP_r}{dr} = -\frac{\kappa\rho}{4\pi c r^2} L(r)$$

relation between the radiative energy flow & the gradient of the radiation pressure.

If the star is static, i.e.:

$$\begin{cases} \dfrac{dL(r)}{dr} = 4\pi r^2 \rho \epsilon & \text{(eq. 4)} \\[2ex] \dfrac{dP}{dr} = -\dfrac{GM(r)}{r^2}\rho & \text{(eq. Hydrostatic eq.)} \end{cases}$$

$$\Rightarrow \quad \frac{dP_r}{dP} = \frac{dP_r/dr}{dP/dr} = \frac{\kappa L(r)}{4\pi c\, G M(r)}$$

Let's define $\eta(r) := \frac{\bar{\epsilon}_r}{\bar{\epsilon}} = \frac{L(r)/M(r)}{L/M}$

w $\bar{\epsilon}_r$ average rate of nuclear energy generation inside r
$\bar{\epsilon}$ average rate of nuclear energy generation for the whole star

$$\Rightarrow \quad \frac{dP_r}{dP} = \frac{L}{4\pi c G M}\,\kappa\eta(r).$$ Integrating

$$P_r(r) = \frac{L}{4\pi c G M}\int_0^{P(r)} \kappa(r)\,\eta(r)\,dP = \frac{L}{4\pi c G M}\,P(r)\,\underbrace{\overline{\kappa\eta}}$$

pressure average of $\kappa\eta$ between $r=r$ & $r=R$.

STROMGREN THEOREM:
the ratio of the radiation pressure to the total pressure $(P_r(r)/P(r))$ at a point inside a star in radiative equilibrium is proportional to the average value of $\kappa\eta$ for the region exterior to the point r, the average being taken w/ respect to dP, where P is the total pressure.

At the center:
$$\boxed{L = \frac{4\pi c\, G M (1-\beta_c)}{\overline{\kappa\eta}} \quad \text{w/ } P_r = (1-\beta)P}\,,$$

w/ $\overline{\kappa\eta}$ average over the entire star

β_c ratio of gas pressure to total pressure @ $r=0$,

LUMINOSITY FORMULA
For stars in radiative equilibrium.

If $\kappa\eta = \text{const} \Rightarrow \beta(r) = \text{const.} \Rightarrow$ the constancy of β in stars in radiative equilibrium depends on the constancy of $\kappa\eta$.

NOTE: κ increases by several orders of magnitude from the center to the surface

η decreases by several orders of magnitude from $r=0$ to $r=R$; $\eta(r=R)=1$.

$\Rightarrow$ fortuitously good resemblance of the standard model ($n=3$) to real stars.

NOTE: Let's assume κ independent of M of stars on main sequence.

For $n=3$ (standard model) $T_c \propto M^{2/3}\bar{\rho}^{1/3}$, $\bar{\rho} \propto M/R^3$

$$\Rightarrow T_c \propto M/R$$

Assuming that $\frac{dT}{dr} \propto \frac{T_c}{R}$

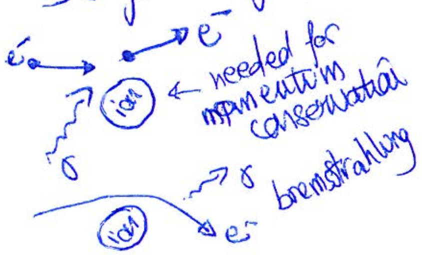
From (5) $\Rightarrow$ $L \propto R^2 \frac{1}{\rho} T^3 \frac{dT}{dr} = R^2 \frac{1}{M/R^3} \frac{M^3}{R^3} \frac{M}{R^2}$

$\Rightarrow$ $\boxed{L \propto M^3}$ not that different from what observed for main-sequence stars.

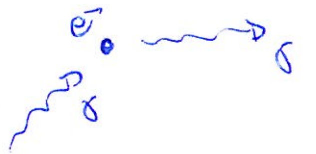
OPACITY

- Evaluation of rate of energy transport by radiative transfer ($L(r)$, eq. 5) requires calculation of κ, Rosseland mean opacity. $\kappa = \kappa(\rho, T)$ for each composition.
- In stars, the radiative opacity is almost entirely due to 4 basic types of event:

1. bound-bound absorption, ie: absorption of a photon by an atom during which a bound e^- makes a transition to a bound state of higher energy. It is a true-absorption process & the inverse is the emission of light by a downward transition.
2. bound-free absorption: absorption of a photon by an atom during which a bound e^- makes a transition to a continuum state, ie: photo-ionization. It is a true-absorption process, & its inverse is radiative recombination.
3. free-free absorption: absorption of a photon by a continuum e^- as it passes an ion & makes a transition to another continuum state of higher energy. It is a true-absorption process, & its inverse is called bremsstrahlung.



4. Scattering from free e^-s: scattering of photons by individual free e^-s in the gas, called Compton scattering, or in the non-relativistic approximation usually in stars, Thomson scattering. It is not true absorption, ie: the scattered photon energy equals the incident energy.



From pag. 8 (Notes 5÷8) $\quad \lambda = \dfrac{1}{n\sigma}$, λ mean free path

From pag. 9c (Notes 1÷4)

$$I_\nu = I_{0,\nu}\, e^{-\int_0^r \kappa_\nu \rho\, dr} = I_{0,\nu}\, e^{-\kappa_\nu \rho r}$$

I_ν: after absorption; $I_{0,\nu}$: before absorption; $I_{0,\nu}e^{-\kappa_\nu\rho r}$: for constant density & opacity κ_ν

ie: the intensity declines exponentially, falling by a factor e^{-1} over a characteristic distance

$$r = \frac{1}{K_\nu \rho} \equiv \lambda_{\text{(mean free path)}}$$

$$\Rightarrow \quad \frac{1}{K_\nu \rho} = \frac{1}{n\sigma} \Rightarrow \boxed{K_\nu = \frac{n\sigma_\nu}{\rho}}$$

ie: the opacity K_ν depends on the number density n of absorber, the density ρ & the cross-section of the interaction $\sigma_\nu = \sigma(\nu)$.

BOUND - BOUND ABSORPTION: no simple equation for bound-bound transitions describing all of the contributions to the opacity by individual spectral lines.

σ_ν : $\dfrac{\text{energy absorbed from an initial state } s \text{ at a frequency } \nu}{\text{incident energy per unit area at a frequency } \nu}$.

$$\sigma_{bb}(\nu) = \frac{4\pi^2 e^2}{hc\, m^2 \nu_{KS}} \left| \langle K | \, e^{\frac{i 2\pi \nu_{KS}}{c} \mathbf{n} \cdot \mathbf{r}} \, \mathbf{p} \cdot \mathbf{e} \, | s \rangle \right|^2 \underbrace{\delta(\nu - \nu_{KS})}$$

delta function, ie: $\int_{-\infty}^{\infty} f(\nu)\,\delta(\nu - \nu_{KS})\, d\nu = f(\nu_{KS})$

so that $\int E(\nu)\sigma(\nu)d\nu$ is the energy absorbed from the pulse in making an atomic excitation between state s K & S

The photons absorbed must have frequency $h\nu_{KS} = E_K - E_S$

Actual atomic lines are not infinitely sharp in energy because the initial & final states of the e^- are not (uncertainty principle)

$$\Rightarrow \int_{\Delta\nu} \sigma(\nu)\, d\nu = \frac{4\pi^2 e^2}{m^2 c\, h\nu_{KS}} \left| \langle K | \, e^{\frac{i 2\pi \nu_{KS}}{c} \mathbf{n} \cdot \mathbf{r}} \, \mathbf{p} \cdot \mathbf{r} \, | s \rangle \right|^2$$

$$= \frac{16\pi^4 e^2}{hc} \nu_{KS} \left| \langle K | \mathbf{r} \cdot \mathbf{e} | s \rangle \right|^2$$

small frequency band containing the line shape

if only electric-dipole transitions are considered

With the states k & s spatially represented by wave functions. (11)
(Schrödinger equation) ...

BOUND – FREE ABSORPTION: ionization process in which the absorption of a photon by an ion leads to the emission of an e^- into a continuum state

$$h\nu_{ks} = \chi + \frac{p^2}{2m}$$

χ: e^- binding energy

$\frac{p^2}{2m}$: kinetic energy upon ejection

$$\ldots \quad \sigma_{bf}(\nu) = \frac{64\pi^4 m e^{10}}{3\sqrt{3}\, c h^6} \frac{Z^4}{n^5} \frac{g(\nu, n, l, Z)}{\nu^3}$$

$$= 2.82 \times 10^{29} \frac{Z^4}{n^5 \nu^3} g(\nu, n, l, Z) \ \text{cm}^2$$

for a hydrogenic electron in a state w/ principal quantum number n & ν photon frequency.

$g(\nu, n, l, Z)$ is the Gaunt factor, which depends on the initial-state quantum numbers n & l & is a slowly varying function of the photon frequency.

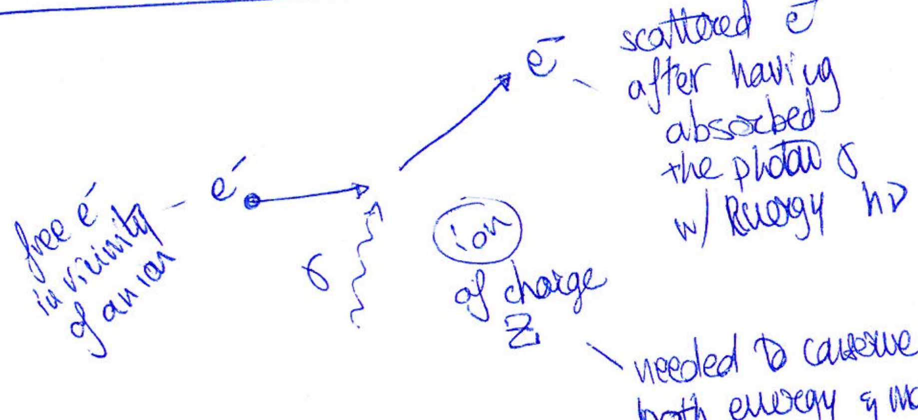
$g(\nu, n, l, Z) \underset{\text{w/in } 20\%}{\approx} 1$ for hydrogen-like atoms.

For hydrogen atom $\sigma_{bf}^{H}(\lambda) \simeq 1.31 \times 10^{-15} \frac{1}{n^5} \left(\frac{\lambda}{500\,\text{nm}}\right)^3 \text{cm}^2$

$$\boxed{\bar{\kappa}_{bf} = 4.34 \times 10^{25}\, g_{bf}\, Z(1+X) \frac{\rho}{T^{3.5}} \propto \kappa_0 \rho / T^{3.5}}$$

w/ $[\rho] = \text{g/cm}^3$ $\qquad$ κ_0 depending on composition

FREE-FREE ABSORPTION (the inverse is BREMSSTRAHLUNG)



$$\frac{p_K^2}{2m} = \frac{p_S^2}{2m} + h\nu$$

$$\overline{\sigma}_{ff} = \frac{16\pi^2 Z^2 e^6 n_e}{3\sqrt{3}\,(2\pi m)^{3/2}(KT)^{1/2}}\,\frac{1}{\nu^3}\int_0^\infty e^{-x}\, g_{ff}(x)\,dx \quad , \text{ w/ } x = \frac{mv^2}{2KT}$$

$\overline{\sigma}_{ff}$: average free-free cross section for photons of frequency ν (since the e's possess a maxwellian velocity distribution)

g_{ff}: Gaunt factor for free-free transition, depending on e^- velocity & energy of the absorbed photon.

$$\overline{g}_{ff}(\nu, Z, T) = \int_0^\infty e^{-x}\, g_{ff}(x,\nu,Z,T)\,dx$$

average free-free Gaunt factor ≈ 1 in almost all astrophysical cases.

The effective photon energy for a free-free ~~absorption~~ is $(h\nu)_{eff} \approx 5.82\,KT$

$$\langle\sigma_{ff}\rangle = 1.25\times10^{-1}\,\frac{Z^2\,\overline{g}_{ff}}{\mu_e}\,\frac{\rho}{T^{3.5}}\ \mathrm{cm}^2$$
$$\simeq 6.25\times10^{-2}\,(1+X)\,Z^2\,\overline{g}_{ff}\,\rho\,T^{-3.5}\ \mathrm{cm}^2$$

$$\Rightarrow \quad \overline{\sigma}_{ff}(Z,\nu,T) = 3.69\times10^{8}\,\frac{Z^2\, n_e\,\overline{g}_{ff}}{T^{1/2}\,\nu^3}\ \mathrm{cm}^2$$

$$\Rightarrow \quad \langle\kappa_{ff}\rangle = 7.53\times10^{22}\,\frac{\rho}{\mu_e T^{3.5}}\sum_Z \frac{Z^2\,\overline{g}_{ff}(Z, h\nu/KT = 7)\,X_Z}{A_Z} \quad \mathrm{cm}^2/\mathrm{g}$$

$\langle\kappa_{ff}\rangle$: Rosseland mean opacity for free-free transition

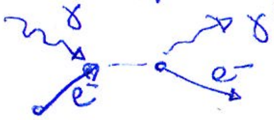
$\propto \kappa_0\,\rho/T^{3.5}$

AKA: Kramers opacity.

$$\Rightarrow \quad \boxed{\overline{\kappa}_{ff} \sim 3.7\times10^{22}\,\underbrace{(X+Y)}_{(1-Z)}\,(1+X)\,\frac{\rho}{T^{3.5}}\,\overline{g}_{ff}\ \frac{\mathrm{cm}^2}{\mathrm{g}}}$$

SCATTERING FROM e^-s:

scattering opacity dominates when temperature is sufficiently high, otherwise bound-free opacity dominates.



⋮

Thomson scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{power radiated per unit solid angle}}{\text{incident power per unit of area}} = \left(\frac{e^2}{mc^2}\right)^2 \frac{1}{2}\left(1+\cos^2\Theta\right)$$

w/ Θ, angle between the direction of observation & the direction of photon propagation

Thomson scattering is not valid for relativistic particles or for photon energies $\approx mc^2$ (ie: when $T > 10^9 K$)

$\frac{d\sigma}{d\Omega} \propto \frac{1}{m^2}$ → free e^-s scattering is much more important than scattering from nuclei (neglected)

↳ The total cross section for scattering of photons into all angles by a free e^-:

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 = 0{,}665 \times 10^{-24}\,\text{cm}^2$$

THOMSON CROSS SECTION

σ_T is independent of frequency → κ_{es} in the Rosseland mean is a constant if scattering opacity is due to free e^-s. Photons scatter from ions & molecules in the cooler outer regions of a star, where ionization is incomplete (Rayleigh scattering, frequency dependent)

For complete ionization

adequate for most problems of stellar interior

$$\kappa_{es} = \frac{0.4}{\mu_e} \approx 0.2\,(1+X_H)\ \frac{\text{cm}^2}{\text{g}}$$

X_H: hydrogen mass fraction

$$\kappa_{es} = \sigma_T \cdot \frac{\#\text{ free } e^-\text{s}}{\text{gram}}$$

- $K_{es} > K_{ff}$ only when $T > 4.5 \times 10^6 \rho^{0.286}$ for an ionized gas composed of H & He.
- when the photon energies become significant fractions of mc^2
  → Klein-Nishina formula for $d\sigma$ & the integrated cross section

$$\sigma = \frac{3}{4}\sigma_T \left\{ \frac{1+\epsilon}{\epsilon^2}\left[\frac{2+2\epsilon}{1+2\epsilon} - \frac{\ln(1+2\epsilon)}{\epsilon}\right] + \frac{\ln(1+2\epsilon)}{2\epsilon} - \frac{1+3\epsilon}{(1+2\epsilon)^2} \right\} \quad w/\ \epsilon = \frac{h\nu}{mc^2}$$

- At high energies, the photons may create $e^+ - e^-$ pair in the field of a nucleus, ie: $\gamma + Z \to Z + e^+ + e^-$ if $h\nu > 2mc^2$

$$\sigma_{pair} \approx Z^2\, 10^{-3}\, \sigma_T \quad \text{for } h\nu > 2mc^2$$

TOTAL RADIATIVE OPACITY

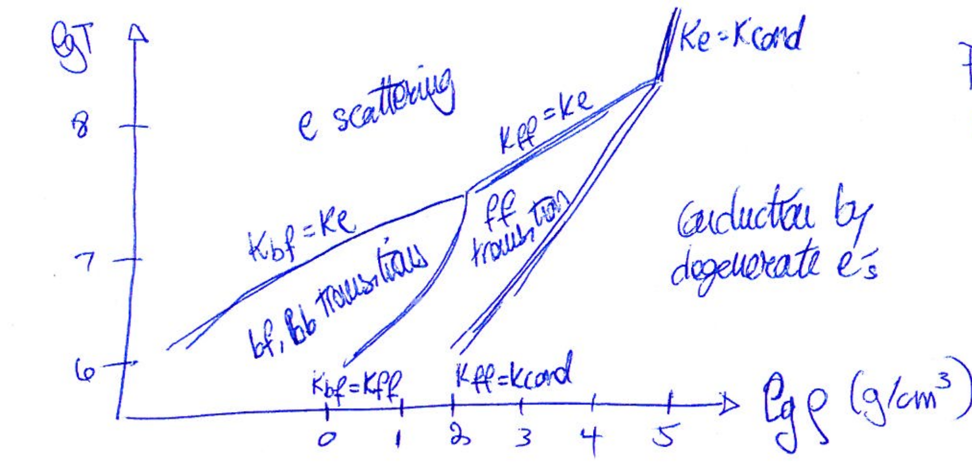
At low T, a significant number of nuclei are only partially ionized
→ opacity dominated by K_{bb} & K_{bf}

As ionization is nearly complete, K_{ff} becomes dominant, but because the Rosseland mean of K_{ff} decreases w/ increasing T, a temperature will be reached where scattering by free e^-s dominates. But all forms of opacity contribute at all time.

$$\Rightarrow \underbrace{\left[K_{bb}(\nu) + K_{bf}(\nu) + K_{ff}(\nu)\right]}_{K_{\nu a}}\left(1 - e^{-h\nu/KT}\right) + \underset{K_{\nu s}}{K_s}$$

to be used in the Rosseland mean opacity $1/K$ formula.

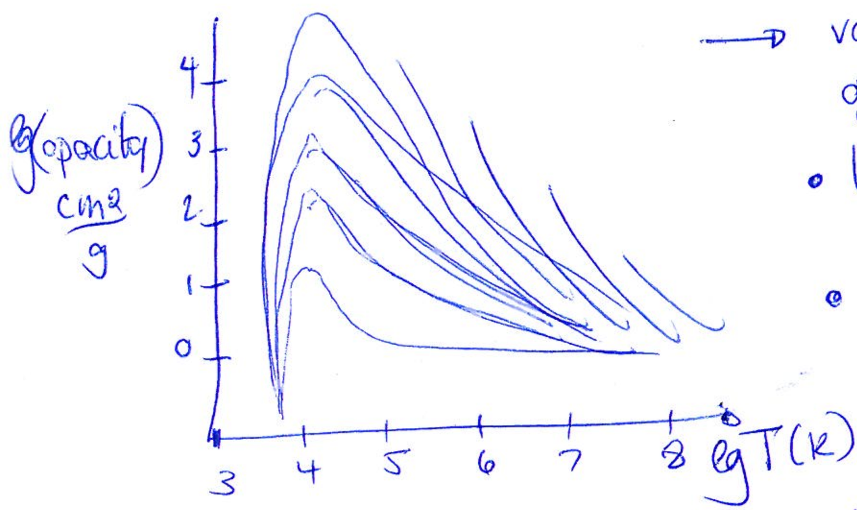
pop I composition
(X=0.7, Y=0.28, Z=0.02)



$K_{es} = 0.2\,(1 + X_H)\ \frac{cm^2}{g}$ e^- scattering dominates at high temperatures

As T decreases, K_{bf} overtakes K_{es} at low densities, & K_{ff} overtakes K_{es} at intermediate densities.

At a given value of temperature (e.g. 3×10^6 K), the dominant mechanisms are e^- scattering at very low density, bound absorption at low density, free-free absorption at intermediate density, & e^- conduction at high density.



→ values of total opacity as a function of T for different values of ρ.

- large peak @ 10^4–10^5 K due to ionization of H & He.
- at low T, there are relatively few photons w/ enough energy for these radiation processes → small Rosseland mean
- At higher T, the ionization reduces the # of bound e's per grain ⇒ bound absorption gets small.

$\boxed{\odot}$ The largest radiative opacities are found in the H & He ionization zones in stars ⇒ the temperature gradients required to radiatively transport energy through these zones are so large that the zones are almost always unstable against convection. Opacity approaches K_{es} at high T.

- The bound absorptions & free-free absorptions are both strongly dependent upon Z, the fraction by mass of heavy elements $\Rightarrow$ true absorption $\propto Z$.
- For a given Z, the opacity depends on the abundance of H relative to He.

From the LUMINOSITY FORMULA $L = \dfrac{4\pi c\, GM(1-\beta_c)}{\overline{\kappa\eta}}$

Within the standard model, $1-\beta = \text{const}$
$\kappa\eta = \text{const}$

If $\kappa = \kappa_0 \rho T^{-3.5}$ (Kramer's opacity law)

$$\Rightarrow \overline{\kappa\eta} = \kappa_0 \eta_c \rho_c T_c^{-3.5} = \kappa_0 \eta_c \frac{a}{3} \frac{\mu m_H}{k} \frac{\beta_c}{1-\beta_c} T_c^{-1/2}$$

for nondegenerate gas

$$P_{gas} = \beta_c P_c = \frac{\rho_c k T_c}{\mu m_H} \to P_c = \frac{\rho_c k T_c}{\mu m_H \beta_c}$$

$$P_{rad} = (1-\beta_c) P_c = \frac{a T_c^4}{3} \to P_c = \frac{a T_c^4}{3(1-\beta_c)}$$

$$\Rightarrow \rho_c = \frac{a T_c^3 \mu m_H \beta_c}{3(1-\beta_c) k}$$

$T_c = 4.6\times 10^6 \mu\beta (M/M_\odot)^{2/3} \rho_c^{1/3}$

$M = 18 \dfrac{\sqrt{1-\beta}}{\mu^2\beta^2} M_\odot$

$$\Rightarrow \boxed{\frac{L}{L_\odot} = 2.7\times 10^{25} \frac{1}{\kappa_0 \eta_e} \left(\frac{M}{M_\odot}\right)^{5.5} \left(\frac{R_\odot}{R}\right)^{0.5} (\mu\beta_c)^{7.5}}$$

Although steeper mass-luminosity relationship than observed ($L/L_\odot \propto (M/M_\odot)^4$)

$\boxed{L \propto \mu^{7.5}/\kappa_0}$, ie: dependence of luminosity on composition.

[6] Chandrasekhar demonstrated that the outer layers of a star in radiative equilibrium w/ Kramers opacity & low radiation pressure are very similar to the standard model w/ $n=3$. The interior of such a star is a polytrope w/ $n=3.25$ if the energy generated ϵ is constant.

Is the energy is liberated all at the center the polytropic index varies continuously from $n=3.25$ at the radiative surface to $n=1.5$ at the convective center.

CONDUCTION

Consider motion of charged particles in an ionized gas.
If gas non degenerate $\longrightarrow$ each particle has average energy

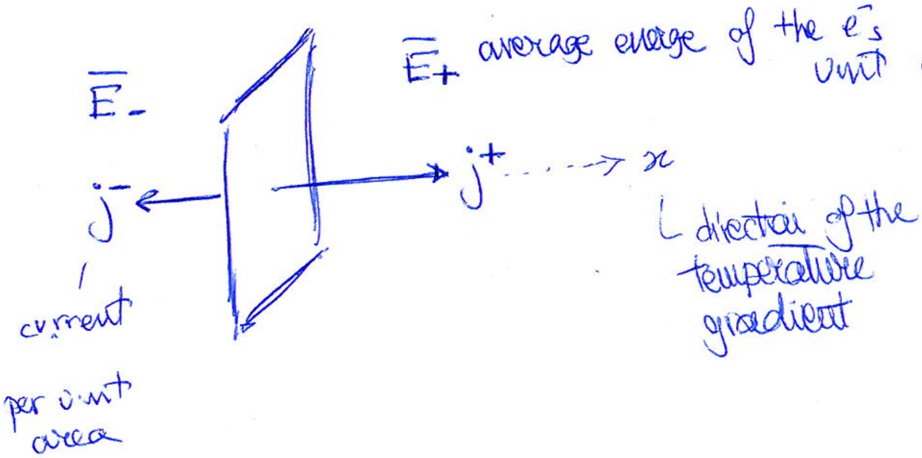
$$E_T = \frac{3}{2} KT \Longleftrightarrow v_T = \left(\frac{2E_T}{m}\right)^{1/2} = \left(\frac{3KT}{m}\right)^{1/2}$$

$\Rightarrow$ e^- velocities are greater than ion velocities by a factor $\frac{v_e}{v_{ion}} = \left(\frac{m_{ion}}{m_e}\right)^{1/2} \approx 43\sqrt{A_{ion}}$

A_{ion} atomic mass

$m_i \approx m_p A_{ion}$

If e^- are degenerate $v_e/v_{ion} >> 43\sqrt{A_{ion}}$, as e^-s are forced into higher momentum states, ie: the e^- gas can be thought as composed of fast-moving e^-s relative to a nearly-stationary background of positive ions.



$\bar{E}_+$ average energy of the e^-s responsible for the current per unit area j_+

$$\frac{\# \text{ of } e^-}{s \text{ cm}^2} = \text{current per unit surface } j^\pm$$

Because of particle conservation $\Rightarrow j_+ = j_-$

Net energy flux $@\, x = j_+\bar{E}_+ - j_-\bar{E}_- \neq 0$ in a stellar interior

NOTE: $\bar{E}_+ \neq \bar{E}_-$; for example, if the e^-s on the left are hotter $\Rightarrow \bar{E}_+ > \bar{E}_-$.

The conduction heat will be proportional to the product of the electron current ($j\pm$) & the excess energy per particle moving in the direction of the heat flow ($\bar{E}_+ - \bar{E}_-$)

j depends only on the density n_e & velocity v_e of e^-s, ie: $j = n_e \cdot v_e$. It is $(\bar{E}_+ - \bar{E}_-)$ which is difficult to estimate in conductive transport.

$\Delta E = \bar{E}_+ - \bar{E}_-$ is roughly proportional to $\frac{\partial E}{\partial r}$ (ie: how steeply the average energy changes w/ position) & $\bar{\ell}$ (ie: the average distance traveled by the e^-s between major collisions). The collisions will transfer energy in a diffusion process. This mechanism of energy transport is usually less efficient than radiative transfer, since photons have larger $\bar{\ell}$, ie: photons can see a bigger temperature drop than e^-s.

$$\Delta E \simeq -\bar{\ell} \cdot \frac{\partial E}{\partial r}$$

$$\Rightarrow \quad Q_r = j_+ \bar{E}_+ - j_- \bar{E}_- = n_e v_e \Delta E = -n_e v_e \bar{\ell} \frac{\partial E}{\partial r}$$

Taking $E = \frac{3}{2} kT$

$$\frac{\partial E}{\partial r} = \frac{3}{2} k \frac{\partial T}{\partial r}$$

$\Rightarrow$ Energy flux due to conduction

$$Q = \underbrace{-\frac{3}{2} k\, n_e v_e \bar{\ell}}_{\text{diffusion coefficient depending on } \bar{\ell},\ v_e \ \&\ \bar{\ell}} \frac{\partial T}{\partial r}$$

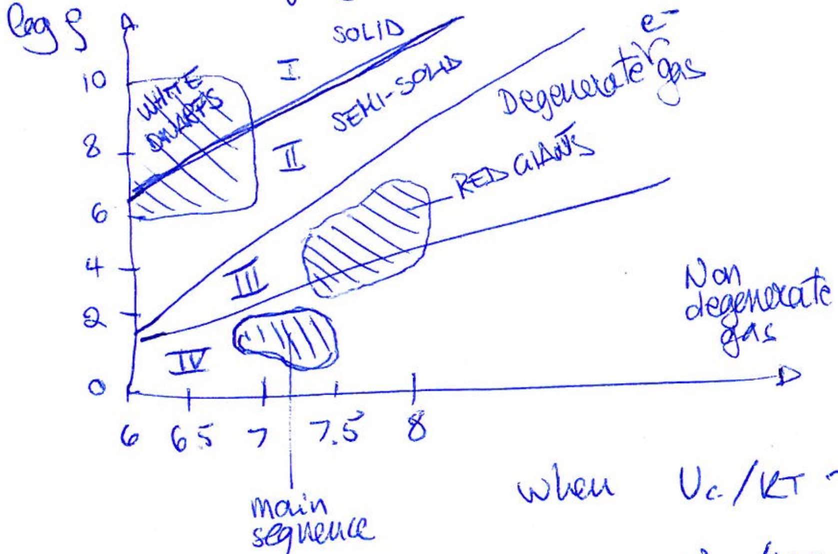
express as a diffusion process

diffusion coefficient depending on $\bar{\ell}$, v_e & $\bar{\ell}$.

$\bar{\ell} = \frac{1}{n_i \sigma}$, w/ n_i ion density, σ cross section of interaction of e^-s w/ ions.

σ needs to be calculated. In principle, $e^- - e^-$ & e^--ion scatterings have to be considered. However $e^- - e^-$ scattering is negligible in the conductivity calculation, & scattering from the ions is the major scattering (Rutherford scattering).

→ see figure $\log\rho$ vs $\log T$ & state of ions.



ρ & T determine the relative importance of the coulomb interaction energy per ion & KT.

when $\frac{U_c}{KT} > 1$ → ions are forced into a lattice structure, ie: SOLID.

when $U_c/KT \sim 1$ → semi-solid

$U_c/KT \lesssim 1$ → e^-s still strongly degenerate, but ions now more as a non-degenerate gas (III)

In IV, both e^-s & ions are non degenerate gases.

Conductivity depends on both the state of the e^-s gas & of the ions gas

Using a very approximate approach

$\sigma \simeq \pi r_0^2$ w/ r_0 typical distance of interaction

$$\underbrace{\frac{p^2}{2m_e}}_{\text{kinetic energy}} = \underbrace{\frac{Z_i e^2}{r_0}}_{\text{electrostatic potential energy}} \longrightarrow r_0 \simeq \frac{2 m_e Z_i e^2}{p^2}$$

$$\Rightarrow \sigma \simeq \frac{\pi Z_i^2 e^4}{(P_e^2/2m_e)^2}$$

$$\Rightarrow Q = -\frac{3}{2} K \underbrace{n_e v_e}_{v_e = p_e/m_e} \frac{1}{n_i \sigma} \frac{\partial T}{\partial r} = -\frac{3}{2} K n_e \frac{p_e^5}{n_i m_e^3} \frac{1}{4\pi Z_i^2 e^4} \frac{\partial T}{\partial r}$$

The conductive heat flow Q can be written as

$$Q = -\frac{4ac}{3\rho\kappa_c} T^3 \frac{\partial T}{\partial r}$$ ← this is useful as it writes the heat-conduction equation in the same form as that of radiative transfer.

Comparison w/ the above expression for Q results in:

⇒ CONDUCTIVE OPACITY
$$\kappa_c = \frac{32\pi}{3^{9/2}} \frac{ac}{\rho} \frac{n_i}{n_e} e^4 Z_i^2 T^{1/2} \frac{m_e^{1/2}}{k^{7/2}}$$

replacing $\frac{p_e^2}{2m_e} = \frac{3}{2}kT$
for non-degenerate e^-s

Substituting $n_e = Z_i n_i \approx \frac{A}{2} n_i$
$A \approx 2Z$, mass number

$$\Rightarrow \boxed{\kappa_c \simeq 5\times10^3 \frac{Z^2}{A} \frac{\sqrt{T/10^7}}{(\rho/10)} \; cm^2/g}$$

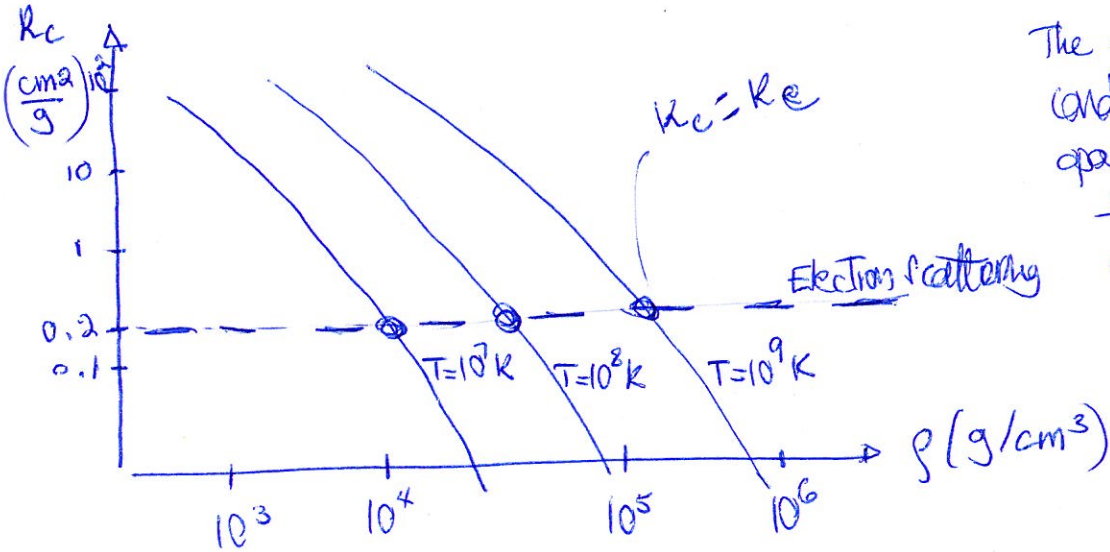
Conduction will be more important than radiative transfer if $\kappa_c < \kappa_r$, although they occur simultaneously.
⇒ the radiative-transfer equation can be made to contain both effects if a generalized opacity is defined by

$$\frac{1}{\kappa} = \frac{1}{\kappa_r} + \frac{1}{\kappa_c}.$$

If $\kappa_c \ll \kappa_r$ ⇒ $\kappa \approx \kappa_c$
$\kappa_c \gg \kappa_r$ ⇒ $\kappa \approx \kappa_r$

NOTE: The opacities represent a "resistance" to heat flow & have an electrical analogy in the resistance to current.

NOTE: The above calculation of κ_c is an approximation. Proper calculation will depend on the state of the matter (weak degenerate, strong degenerate, or intermediate degeneracy).



The density at which the conductive opacity = radiative opacity due to scattering from the electrons increases ~ one order of magnitude for each order-of-magnitude increase in the temperature.

↳ in case of degeneracy, $\bar{\ell}$ becomes very large (ie: the e^-s do not interact w/ matter)

$\Rightarrow$ κ_c becomes very small, ie: the conduction becomes important.

For He, $T = 10^8 K$, $\rho = 10^3 g/cm^3$ (non-degenerate gas)

↳ $A = 4$ $\Rightarrow$ $\kappa_c \simeq 170 \frac{cm^2}{g} \gg \kappa_r$

$Z = 2$

More generally $\kappa_c = \dfrac{4acT^3}{3\rho\lambda_c}$

$\lambda_c \approx \dfrac{16\sqrt{2}}{\pi^{3/2}} \dfrac{k^{7/2}T^{5/2}}{m_e^{1/2} e^4 Z^2}$ $\longrightarrow$ $\kappa_c \approx 1.2\times10^3 \, Z^2 \dfrac{\sqrt{T/10^7}}{\rho/10}$ cm^2/g

weak degeneracy (maxwellian case)

$\lambda_c \simeq 2.36\times10^3 \dfrac{1}{\mu_e}\dfrac{\rho T}{Z}$ (strong degeneracy, non relativistic) $\Rightarrow$ $\kappa_c = 1.28\times10^{-7} \dfrac{\mu_e Z T^2}{\rho^2}$

CONVECTIVE INSTABILITY

In the diffuse mechanisms of energy transport, the heat flux is

$$H = -\frac{4ac}{3\kappa\rho} T^3 \frac{dT}{dr} \;, \text{ ie: } \; H \propto \frac{dT}{dr}$$

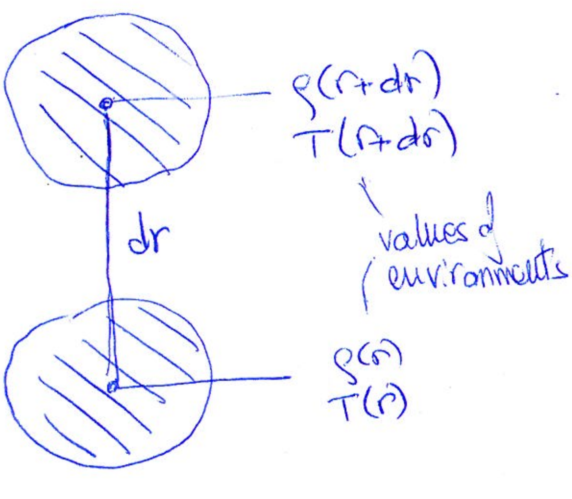
$$H \propto \frac{1}{\kappa} \leftarrow \text{total opacity}$$

If stellar structure $\longleftrightarrow$ polytrope $n=3$ (standard model) $\Rightarrow \frac{T^3}{\rho} = \text{const.}$

$$\Rightarrow \text{flux} \propto \frac{1}{\kappa} \frac{dT}{dr}$$

In a static model, the heat flux must be sufficiently large to carry out all the energy liberated within a given sphere, & this requirement establishes the temperature gradient dT/dr.

<u>IF</u> dT/dr is too great $\Rightarrow$ Instability to convective gas motion.
(Schwarzschild 1906)



The mass element is displaced by dr without exchanging heat with the environment (ie: adiabatic change)

$$\Rightarrow \frac{dP}{P} = -\Gamma_1 \frac{dV}{V} = \Gamma_1 \frac{d\rho}{\rho}$$

$$\text{w/ } \; dP = P(r+dr) - P(r) = \left(\frac{dP}{dr}\right) dr$$

$$\Rightarrow \rho^* = \rho(r) + (d\rho)_s = \rho(r) + \frac{1}{\Gamma_1} \frac{\rho}{P} \left(\frac{dP}{dr}\right) dr$$

expanded density

$$\frac{dP}{P} = \Gamma_1 \frac{d\rho}{\rho}$$

$$dP = \left(\frac{dP}{dr}\right) dr$$

IF $\rho^* > \rho(r+dr) \Rightarrow$ displaced element is denser than environment & will settle back under gravity

$\rho^* < \rho(r+dr) \Rightarrow$ net bouyant force $\rightarrow$ continue upward stability condition : $\begin{cases} \text{stable} & \text{if } \rho^* > \rho(r+dr) \\ \text{unstable} & \text{if } \rho^* < \rho(r+dr) \end{cases}$

Since $\rho(r+dr) = \rho(r) + \left(\frac{d\rho}{dr}\right) dr$

$\Rightarrow$ $\boxed{\frac{1}{\Gamma_1} \frac{\rho}{P} \frac{dP}{dr} > \frac{d\rho}{dr}}$ stability condition

In terms of temperature, $T^* < T(r+dr)$ [stable]

$\Rightarrow$ $\left| \left(\frac{dT}{dr}\right)_{star} \right| < \left| \underbrace{\left(\frac{dT}{dr}\right)_{adiabatic}}_{\text{adiabatic temperature gradient}} \right|$ stability condition

$\Leftrightarrow$ $\boxed{\left(\frac{dT}{dr}\right)_{star} > \left(1 - \frac{1}{\Gamma_2}\right) \frac{T}{P} \left(\frac{dP}{dr}\right)_{star}}$ algebraic condition for stability

(negative gradients)

ie: if the temperature changes too rapidly w/ distance, instability against convection exists.

From eq. 5 $\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa\rho}{T^3} \frac{L(r)}{4\pi r^2}$

$\Rightarrow$ the stability condition becomes $-\frac{3}{4ac} \frac{\kappa\rho}{T^3} \frac{L(r)}{4\pi r^2} > \left(1 - \frac{1}{\Gamma_2}\right) \frac{T}{P} \frac{dP}{dr}$

Using hydrostatic equilibrium $\frac{dP}{dr} = -\frac{GM_r}{r^2} \rho$

$\Rightarrow$ $\boxed{L(r) \leq \frac{16\pi\, acG}{3\kappa} \left(1 - \frac{1}{\Gamma_2}\right) \frac{T^4}{P} M(r)}$ stability condition

If the luminosity required to maintain energy balance is larger than $\frac{16\pi a c G}{3\kappa}\left(1-\frac{1}{\Gamma_2}\right)\frac{T^4}{P}M(r)$, the energy will have to be carried by <u>convective transport</u>.

<u>IF</u> ideal non-degenerate gas, $L(r) \leq 1.22\times10^{-18}\frac{\mu T^3}{\kappa\rho}M(r)\ \frac{erg}{s}$
$(\Gamma_2 = 5/3)$

if ~~standard~~ standard model $(n=3)$ is correct $\Rightarrow T^3/\rho \sim$ const

$\Rightarrow L(r) \leq$ number proportional to $M(r)/\kappa$

In the outer layers, $M(r)$ is nearly constant

$\Rightarrow L(r) \leq$ number proportional to $1/\kappa$

In the ionization zones (H & He), κ is very large

$\Rightarrow$ the luminosity there exceeds the upper limit for radiative equilibrium

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If the medium is unstable $\rightarrow$ an adiabatically rising element is less dense & hotter than the environment

$\downarrow$

Since the rising mass is hotter than the environment, heat will leak from it into the surroundings

$\leftarrow$ the net effect is the transport of heat to material at a lower temperature

$\downarrow$

<u>energy transport</u> (heating the outer regions & cooling the inner regions)

Mixing-length model $\longleftrightarrow$ model for convective energy transport

the mass element is rising ~~or~~ falling adiabatically for a distance ℓ, the mixing length. After traveling ℓ, the mass element thermalizes w/ the local environment.
After rising a distance ℓ adiabatically, a mass element is hotter than the surrounding by $\Delta T = \left(\left|\frac{dT}{dr}\right| - \left|\frac{dT}{dr}\right|_{adiab}\right)\ell = \ell \underbrace{\Delta \nabla T}_{\text{gradient}}$

The mass element thermalizes at constant pressure, releasing an amount of heat per unit mass $\Delta Q = c_p \Delta T = c_p \ell \Delta \nabla T$
If $\bar{v}$ is the average velocity of the cells, the average excess heat flux is $\underbrace{H}_{\frac{erg}{cm^2 s}} = \rho \bar{v} \Delta Q = \rho \bar{v} c_p \ell \Delta \nabla T$

We need to calculate $\bar{v}$: $\Delta \rho = \ell \left(\left|\frac{d\rho}{dr}\right| - \left|\frac{d\rho}{dr}\right|_{adiab}\right) = \ell \Delta \nabla \rho$

$\Rightarrow \overline{\Delta \rho} = \frac{1}{2} \ell \Delta \nabla \rho$

average density deficiency in the rising element

Average bouyant force per unit of volume $\bar{F} = g \overline{\Delta \rho} = \frac{1}{2} g \ell \Delta \nabla \rho$
w/ $g = GM/r^2$ is the local gravity

$\Rightarrow$ radial acceleration caused by $\bar{F}$: $a = \frac{\bar{F}}{\rho} = \frac{g \ell}{2\rho} \Delta \nabla \rho$

This acceleration produces a final velocity $v = \sqrt{2a\ell} \approx 2\bar{v}$

$\Rightarrow \bar{v} = \frac{1}{2}\left(\frac{g \ell^2}{\rho} \Delta \nabla \rho\right)^{1/2} = \frac{1}{2}\left(\frac{GM}{\rho r^2} \Delta \nabla \rho\right)^{1/2} \ell$

For an ideal monoatomic non-degenerate gas (convection occurs only in non-degenerate regions) $\Rightarrow \Delta\nabla\rho = \frac{\rho}{T}\Delta\nabla T$

$$\Rightarrow \boxed{H = c_p\,\rho\left(\frac{GM}{r^2T}\right)^{1/2}(\Delta\nabla T)^{3/2}\,\ell^2/2}$$

i.e.: the heat flux is proportional to the excess of the temperature gradient to the 3/2 power & to ℓ^2.

The cells will not dissipate until they have moved enough for P & ρ have changed significantly, i.e.: $\sim 10^9$ cm in a star $\Rightarrow \ell \sim 10^9$ cm.

$\Rightarrow$ very small values of $\Delta\nabla T$ are required for normal fluxes.

NOTE: Whenever convection occurs inside a star, the temperature gradient is very nearly the adiabatic temperature gradient.

For a prescription of stellar models, one has to compare dT/dr required to transport the flux radiatively with $(dT/dr)_{adiab}$ at each point. If $\left|\frac{dT}{dr}\right|_{adiab} > \left|\frac{dT}{dr}\right| \Rightarrow$ layer is stable, otherwise the flux is carried convectively along $\frac{dT}{dr} = \left(1-\frac{1}{\Gamma_2}\right)\frac{T}{P}\frac{dP}{dr}$.

This is a good approximation in the interior of a star, but not in the outer layers, where convection so often occurs, because P & ρ change so rapidly compared to their small values. Since $H \propto \rho\ell^2(\Delta\nabla T)^{3/2}$, small ρ & ℓ demand very large $\Delta\nabla T$, i.e.: no longer adequate to use adiabatic temperature gradient to compute the structure of the outer layers. Exact dT/dr is needed to compute

R (& hence T_{eff}) $\Rightarrow$ inability to calculate stellar radii w/ precision for those stars having surface convection zones, ie: low-mass main-sequence stars.

Convection in a star will be turbulent, accompanied by continuous exchange of energy via photons, microturbulence, hydromagnetic waves, & viscous interactions of one mass element with another. $\Rightarrow$ complicated.