Electromagnetic back-reaction from currents on a static straight string

Jeremy Wachter

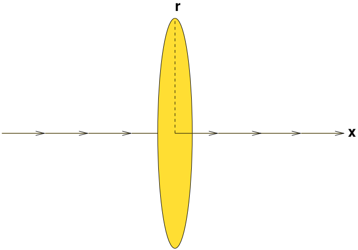
Institute of Cosmology, Department of Physics and Astronomy, Tufts University

05 February 2014

Thanks and acknowledgements

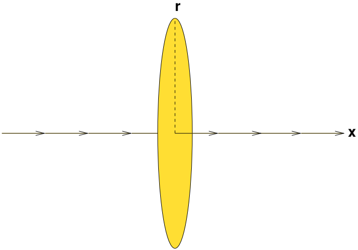
- Ken Olum, for collaboration
- Alex Vilenkin, for suggesting the simpler EM case
- Tanmay Vachaspati, for the solution with rigid charges moving at the speed of light

The Paradox



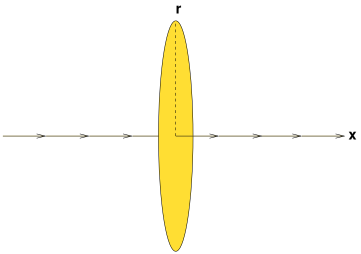
- The electric field goes as $1/r$

The Paradox



- The electric field goes as $1/r$
- The energy density goes as $1/r^2$

The Paradox



- The electric field goes as $1/r$
- The energy density goes as $1/r^2$
- Thus, the energy goes as

$$\int \frac{1}{r} dr = \ln(r)$$

Basic equations

The electromagnetic potential:

$$A^{\mu}(\mathrm{x}) = \int G(X, X')j^{\mu}(X'))d^4X' = \int G(x, t, x', t')J^{\mu}(x', t')dx'dt'$$

(where $X = (t, x, y, z)$, j^{μ} is volumetric current density, J^{μ} is linear current density)

Basic equations

The electromagnetic potential:

$$A^{\mu}(\mathrm{x}) = \int G(X, X')j^{\mu}(X'))d^4X' = \int G(x, t, x', t')J^{\mu}(x', t')dx'dt'$$

(where $X = (t, x, y, z)$, j^{μ} is volumetric current density, J^{μ} is linear current density)

The electric field on the string:

$$E_x(x, t) = F^{10} = -\partial_x A^t - \partial_t A^x$$

Basic equations

The electromagnetic potential:

$$A^\mu(\mathrm{x}) = \int G(X, X')j^\mu(X'))d^4X' = \int G(x, t, x', t')J^\mu(x', t')dx'dt'$$

(where $X = (t, x, y, z)$, j^μ is volumetric current density, J^μ is linear current density)

The electric field on the string:

$$E_x(x, t) = F^{10} = -\partial_x A^t - \partial_t A^x$$

Combine and integrate by parts:

$$E(x, t) = -\int G(x, t, x', t') \left(\partial_{x'} J^t(x', t') + \partial_{t'} J^x(x', t')\right) dx'dt'$$

The integral equation in (x, t)

The current on a superconducting string due to E_x:

$$\partial_x J^t + \partial_t J^x = q^2 E_x$$

The integral equation in (x, t)

The current on a superconducting string due to E_x:

$$\partial_x J^t + \partial_t J^x = q^2 E_x$$

Combine with:

$$E(x,t) = -\int G(x,t,x',t') \left(\partial_{x'} J^t(x',t') + \partial_{t'} J^x(x',t')\right) dx' dt'$$

The integral equation in (x, t)

The current on a superconducting string due to E_x:

$$\partial_x J^t + \partial_t J^x = q^2 E_x$$

Combine with:

$$E(x,t) = -\int G(x,t,x',t') \left(\partial_{x'} J^t(x',t') + \partial_{t'} J^x(x',t')\right) dx' dt'$$

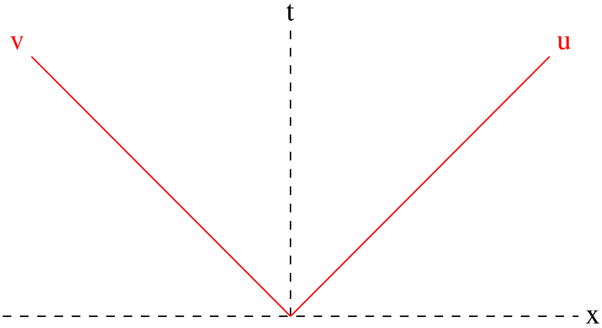
To get the integral equation:

$$E(x,t) = E_{\text{ext}} - Q \int G(x,t,x',t') E(x',t') dx' dt'$$

Here, $E_{\text{ext}} = \delta(x)\delta(t)$ and $Q = q^2$.

Rotating into (u, v)

In the coordinates



the retarded Green function in two dimensions is given by

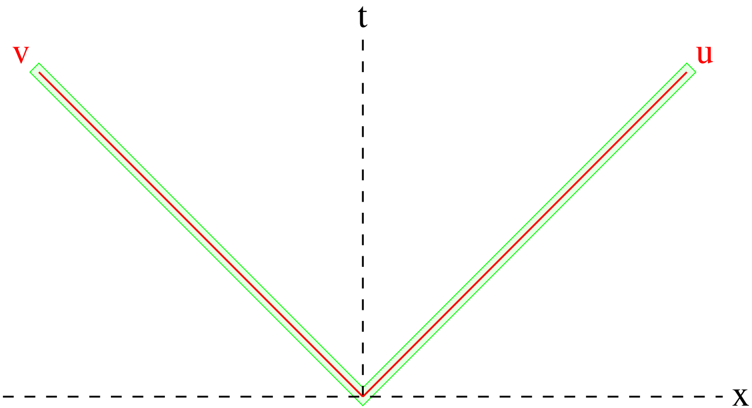
$$G = \frac{\delta(v - v')}{u - u'} + \frac{\delta(u - u')}{v - v'}$$

The integral equation in (u, v)

$$E(u,v) = \delta(u)\delta(v) - Q\left(\int_0^u \frac{E(u',v)}{u-u'+\delta}du' + \int_0^v \frac{E(u,v')}{v-v'+\delta}dv'\right)$$

(where $\delta \approx 10^{-29}$ cm is the string thickness, added to prevent divergence)

The null lines



The electric field on the null lines

With $E_R(u, v) = -\delta(v) f(u)$:

The electric field on the null lines

With $E_R(u,v) = -\delta(v)f(u)$:

$$E(u,v) = \delta(u)\delta(v) - Q\left(\int_0^u \frac{E(u',v)}{u-u'+\delta}du' + \int_0^v \frac{E(u,v')}{v-v'+\delta}dv'\right)$$

The electric field on the null lines

With $E_R(u,v) = -\delta(v) f(u)$:

$$E(u,v) = \delta(u)\delta(v) - Q\left(\int_0^u \frac{E(u',v)}{u-u'+\delta}du' + \int_0^v \frac{E(u,v')}{v-v'+\delta}dv'\right)$$

becomes

$$f(u) = -Q\left(\frac{1}{u+\delta} + \int_0^u \frac{f(u')}{u-u'+\delta}du'\right)$$

The electric field on the null lines

Laplace transform:

$$f(u) = \frac{Q}{\delta} \int_0^\infty \frac{e^{-\alpha(u/\delta+1)}}{(1 - Qe^{-\alpha}\,\mathrm{Ei}(\alpha))^2 + (Q\pi e^{-\alpha})^2} d\alpha$$

The electric field on the null lines

Laplace transform:

$$f(u) = \frac{Q}{\delta} \int_0^\infty \frac{e^{-\alpha(u/\delta+1)}}{(1 - Qe^{-\alpha}\,\mathrm{Ei}(\alpha))^2 + (Q\pi e^{-\alpha})^2} d\alpha$$

Approximate: $u \gg \delta$, $\ln(u/\delta) \gg 1$:

$$f(u) = \frac{Q}{\delta} \int_0^\infty \frac{e^{-\alpha u/\delta}}{(1 + Q(\ln(u/\delta))^2} d\alpha$$

The electric field on the null lines

Laplace transform:

$$f(u) = \frac{Q}{\delta} \int_0^\infty \frac{e^{-\alpha(u/\delta+1)}}{(1 - Qe^{-\alpha}\,\mathrm{Ei}(\alpha))^2 + (Q\pi e^{-\alpha})^2} d\alpha$$

Approximate: $u \gg \delta$, $\ln(u/\delta) \gg 1$:

$$f(u) = \frac{Q}{\delta} \int_0^\infty \frac{e^{-\alpha u/\delta}}{(1 + Q(\ln(u/\delta))^2} d\alpha$$

Solve to get:

$$f(u) = \frac{Q}{u} \frac{1}{(1 + Q\ln(u/\delta))^2}$$

The electric current on the null lines

Recalling:

$$\partial_x J^t + \partial_t J^x = q^2 E$$

The electric current on the null lines

Recalling:

$$\partial_x J^t + \partial_t J^x = q^2 E$$

The current is given by:

$$J^u_R(u, v) = -\frac{Q}{2}\delta(v) \int_0^u f(u')du'$$

The electric current on the null lines

Recalling:

$$\partial_x J^t + \partial_t J^x = q^2 E$$

The current is given by:

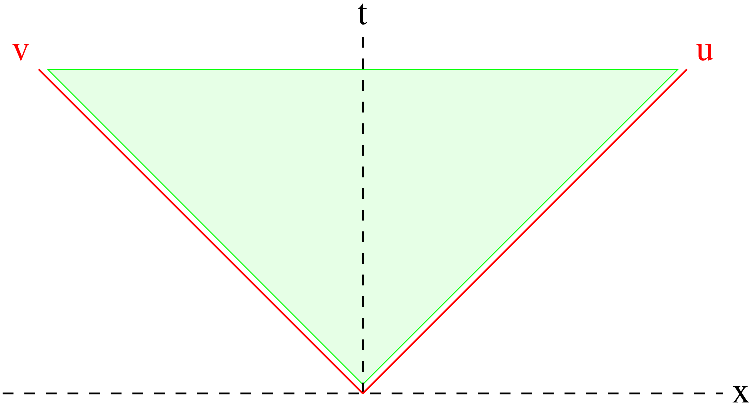
$$J_R^u(u,v) = -\frac{Q}{2}\delta(v)\int_0^u f(u')du'$$

Solve to get:

$$J_R^u(u,v) = \frac{Q}{2}\frac{\delta(v)}{1+Q\ln(u/\delta)}$$

There is significant decline in current once $u = \delta e^{1/Q}$.

The interior



The electric field in the interior

On the interior:

$$E(u, v) = E_R(u, v) + E_L(u, v) + E_{\text{int}}(u, v)$$

The electric field in the interior

On the interior:

$$E(u,v) = E_R(u,v) + E_L(u,v) + E_{\text{int}}(u,v)$$

The integral equation is:

$$E_{\text{int}}(u,v) = Q\left(\frac{f(u)}{v+\delta} + \frac{f(v)}{u+\delta} - \int_0^u \frac{E_{\text{int}}(u',v)}{u-u'+\delta}du' - \int_0^v \frac{E_{\text{int}}(u,v')}{v-v'+\delta}dv'\right)$$

The electric field in the interior

On the interior:

$$E(u,v) = E_R(u,v) + E_L(u,v) + E_{\text{int}}(u,v)$$

The integral equation is:

$$E_{\text{int}}(u,v) = Q\left(\frac{f(u)}{v+\delta} + \frac{f(v)}{u+\delta} - \int_0^u \frac{E_{\text{int}}(u',v)}{u-u'+\delta}du' - \int_0^v \frac{E_{\text{int}}(u,v')}{v-v'+\delta}dv'\right)$$

Laplace transforms:

$$E_{\text{int}}(u,v) = \frac{2Q^2}{uv}\frac{1}{(1+Q\ln(uv/\delta^2))^3}$$

The electric current in the interior

Now, the current is given by

$$J^x_{\text{int}}(u,v) = \frac{Q}{2}\left(\int_0^u E(u',v)du' + \int_0^v E(u,v')dv'\right)$$

The electric current in the interior

Now, the current is given by

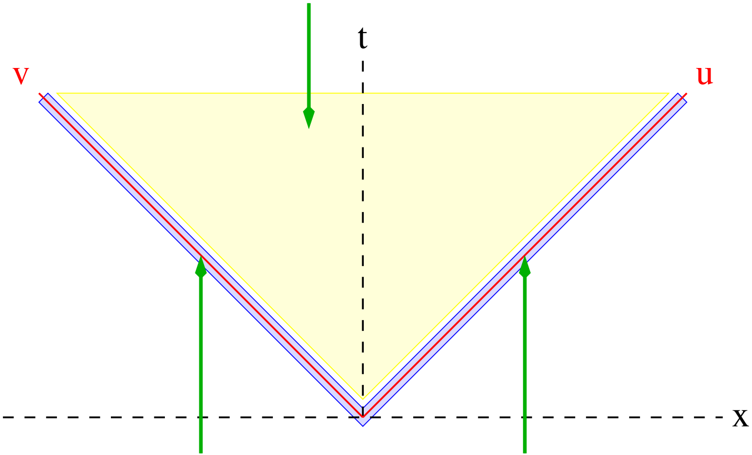
$$J^x_{\text{int}}(u,v) = \frac{Q}{2}\left(\int_0^u E(u',v)du' + \int_0^v E(u,v')dv'\right)$$

which solves to

$$J^x_{\text{int}}(u,v) = -\frac{Q^2}{2}\left(\frac{1}{v}+\frac{1}{u}\right)\frac{1}{(1+Q\ln(uv/\delta^2))^2}$$

Results

$$J^x_{\text{int}}(u,v) = -\frac{Q^2}{2}\left(\frac{1}{v}+\frac{1}{u}\right)\frac{1}{(1+Q\ln(uv/\delta^2))^2}$$



$$J^v_L(u,v) = -\frac{Q}{2}\frac{\delta(u)}{1+Q\ln(v/\delta)} \qquad J^u_R(u,v) = \frac{Q}{2}\frac{\delta(v)}{1+Q\ln(u/\delta)}$$

Results

- The currents decline exactly as quickly as required to cure the divergence

Results

- The currents decline exactly as quickly as required to cure the divergence
- The current reaches half its initial strength at $\delta e^{1/Q}$

Results

- The currents decline exactly as quickly as required to cure the divergence
- The current reaches half its initial strength at $\delta e^{1/Q}$
- For regular electromagnetism ($q^2 = 1/137$), with a string size of $\delta \approx 10^{-29}$ cm, the current declines to half its initial value for the size of the observable universe.

Results

- The currents decline exactly as quickly as required to cure the divergence
- The current reaches half its initial strength at $\delta e^{1/Q}$
- For regular electromagnetism ($q^2 = 1/137$), with a string size of $\delta \approx 10^{-29}$ cm, the current declines to half its initial value for the size of the observable universe.
  - The current declines to three-quarters its initial strength at about 1 Å

Results

- The currents decline exactly as quickly as required to cure the divergence
- The current reaches half its initial strength at $\delta e^{1/Q}$
- For regular electromagnetism ($q^2 = 1/137$), with a string size of $\delta \approx 10^{-29}$ cm, the current declines to half its initial value for the size of the observable universe.
  - The current declines to three-quarters its initial strength at about 1 Å
- For the gravitational case, this same paradox exists (and presumably, the same solution applies)

Results

- The currents decline exactly as quickly as required to cure the divergence
- The current reaches half its initial strength at $\delta e^{1/Q}$
- For regular electromagnetism ($q^2 = 1/137$), with a string size of $\delta \approx 10^{-29}$ cm, the current declines to half its initial value for the size of the observable universe.
    - The current declines to three-quarters its initial strength at about 1 Å
- For the gravitational case, this same paradox exists (and presumably, the same solution applies)
    - With q^2 analogous to $G\mu \approx 10^{-8}$, we will effectively never see this effect

Questions?