KINETIC THEORY OF STRINGS

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Outline

Introduction

Nambu-Goto Strings

Transport Equation

Conservation Equations

Summary

MOTIVATIONS

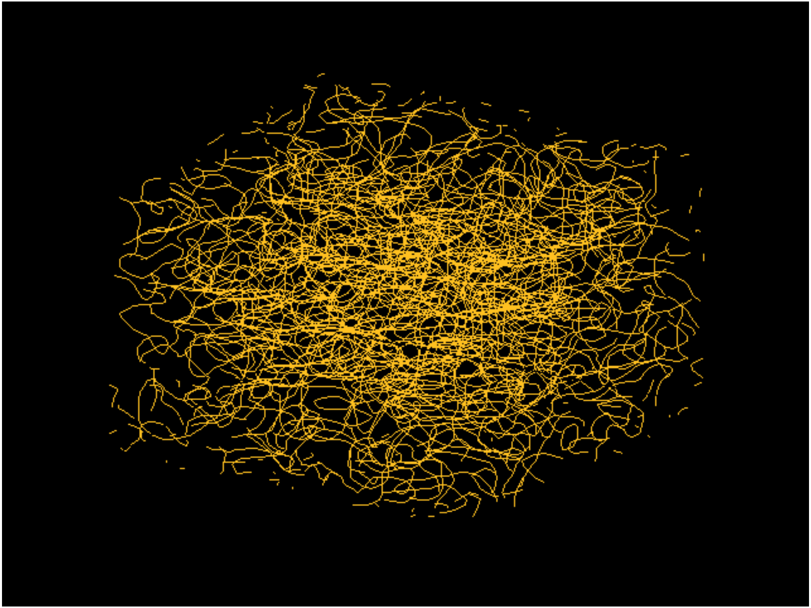
Consider a large network of one-dimensional objects such as

- *cosmic strings after phase transition*
- *fundamental strings near Hagedorn temperature*
- *topological strings in nematic liquid crystal*
- *chain molecules in a polymer, etc.*

If the system is not too large one can study it using simulations. But what if the number of relevant d.o.f. is $10^{10^{10\cdots}}$?

The standard approach would be to use equilibrium SM, but

- *system might be out of equilibrium*
- *ergodic hypothesis might be violated*
- *dynamics might be non-Hamiltonian*
- *partition function might diverge, etc.*



There are at least three more (not unrelated) options:

- Kinetic theory → this talk :)

  [Barrabes, Israel, 1987; Lowe, Thorlacius, 1995; V.V. 2011; V.V. 2013; Schubring, V.V. 2013]

  - *Derive a transport equation for distribution of string segments by considering the dynamics and interactions of individual strings.*
  - *Solve the equation to study the evolution of the systems towards an equilibrium or a steady state which may or may not be unique.*

- Fluid mechanics → next talk :)

  [Stachel, 1980; Carter, 1990; V.V. 2013; Schubring, V.V. 2013]

  - *Coarse-grain the network of strings and treat it as fluid described by fields such as energy density, velocity, tangent vector, etc.*
  - *Derive inviscid fluid eqs. for these fields by considering flows of conserved quantities such as energy, momentum, tangent vectors.*

- Field theory → no talk :(

  [Carter, 1989; Kopczynski, 1989; Schubring, V.V. 2014]

  - *Construct a Lagrangian describing the fluid and add interactions to describe higher order effects of the transport equation.*
  - *Drive the modified conservation equation (for viscous fluid) by considering the diffeomorphism invariance of the Lagrangian.*

COORDINATES FOR A SINGLE STRING

Consider a single world-sheet described by Nambu-Goto action

$$S_{NG} = -\int \sqrt{-\det(h_{ab})}\, d^2\zeta \tag{1}$$

where $h_{ab} \equiv g_{\mu\nu} \frac{\partial x^\mu}{\partial \zeta^a} \frac{\partial x^\nu}{\partial \zeta^b}$ (with tension set to unity $T = \frac{1}{2\pi\alpha'} = 1$.)
In light-cone coordinates ζ^1, ζ^2 the EOM:

$$\boxed{\mathcal{A}^\mu \nabla_\mu \mathcal{B}^\nu = 0} \qquad \boxed{\mathcal{B}^\mu \nabla_\mu \mathcal{A}^\nu = 0} \tag{2}$$

where $\mathcal{A}^\mu \equiv \frac{\partial x^\mu}{\partial \zeta^1}$, $\mathcal{B}^\mu \equiv \frac{\partial x^\mu}{\partial \zeta^2}$ are coord. basis vectors since $[\mathcal{A}, \mathcal{B}] = 0$.
The energy momentum tensor

$$T_{\mu\nu}\sqrt{-g} \equiv \int \tilde{T}^{\mu\nu} \delta^{(4)}(y^\sigma - x^\sigma) \tag{3}$$

is then

$$\boxed{\tilde{T}^{\mu\nu} = h^{ab} \frac{\partial x^\mu}{\partial \zeta^a} \frac{\partial x^\nu}{\partial \zeta^b} \sqrt{-\det(h_{ab})}\, d^2\zeta = 2\mathcal{A}^{(\mu}\mathcal{B}^{\nu)}\, d^2\zeta}. \tag{4}$$

Coordinates for a Network of Strings

Remaining gauge freedom can be removed by defining

$$A^\mu \equiv \frac{\mathcal{A}^\mu}{\mathcal{A}^0} \qquad B^\mu \equiv \frac{\mathcal{B}^\mu}{\mathcal{B}^0}, \tag{5}$$

$$v^\mu \equiv \frac{1}{2}(B^\mu + A^\mu) \qquad \epsilon u^\mu \equiv \epsilon \frac{1}{2}(B^\mu - A^\mu) \tag{6}$$

and demanding that v^μ and ϵu^μ are coord. basis vectors (for τ and σ)

$$[v, \epsilon u] = 0. \tag{7}$$

Then the equations of motion become

$$\boxed{B^\lambda \nabla_\lambda A^\nu = \frac{\dot{\epsilon}}{\epsilon} A^\nu} \qquad \boxed{A^\lambda \nabla_\lambda B^\nu = \frac{\dot{\epsilon}}{\epsilon} B^\nu} \tag{8}$$

where $\frac{\partial \epsilon}{\partial \tau} = \dot{\epsilon} \equiv v^\mu \partial_\mu \epsilon$ and $\frac{\dot{\epsilon}}{\epsilon} = -\Gamma^0_{\lambda\mu} B^\lambda A^\mu$. Then

$$\boxed{\tilde{T}^{\mu\nu} = (v^\mu v^\nu - u^\mu u^\nu)\, \epsilon\, d\tau d\sigma = A^{(\mu} B^{\nu)}\, \epsilon\, d\tau d\sigma = A^{(\mu} B^{\nu)} \tilde{T}^{00}} \tag{9}$$

Conservation Equations

Given a network of many interacting strings, the energy density can be used to form coarse-grained fields such as

$$T^{\mu\nu} = \langle A^{(\mu} B^{\nu)} \rangle \qquad F^{\mu\nu} = \langle A^{[\mu} B^{\nu]} \rangle \tag{10}$$

One can show [Schubring, VV 2013] that

$$\boxed{\nabla_\nu T^{\mu\nu} = 0} \qquad \boxed{\nabla_\nu F^{\mu\nu} = 0} \tag{11}$$

For example, $\nabla_\nu F^{0\nu} = 0$ implies continuity of strings:

$$\nabla \cdot \langle \mathbf{u} \rangle = 0 \tag{12}$$

The two conservation equations can also be rewritten as

$$\boxed{\nabla_\nu \langle A^\mu B^\nu \rangle = 0} \qquad \boxed{\nabla_\nu \langle B^\mu A^\nu \rangle = 0} \tag{13}$$

Note that these equations are exact since no assumptions were made.

Distribution Function

- Conservation equations constrain the dynamics of string fluid but do not describe evolution towards equilibrium.
- In kinetic theory [VV 2013] one first derives a transport equation for distribution $f(A, B, x)$ defined on $\Omega^2 \times \mathcal{M} \equiv S^2 \times S^2 \times \mathcal{M}$.
- Then coarse-grained quantities are given by

$$\langle Q \rangle(x) = \int Q(A, B) f(A, B, x)\, d\Omega^2. \tag{14}$$

- Energy density

$$\rho(x) = \langle 1 \rangle \tag{15}$$

- Probability that random (A, B) rays are in a set $X \subseteq \Omega$

$$p[(A, B) \in X] = \frac{1}{\rho} \int_X f(A, B) d\Omega^2 \tag{16}$$

- For homogeneous eq. we make *string chaos assumption* [VV 2011]

$$p(A_1, B_1 | A_2, B_2 | ...) \approx p(A_1, B_1) p(A_2, B_2) ... \tag{17}$$

that can be justified both analytically [Schubring, VV 2013] and numerically [Balma, Schubring, VV 2014].

Longitudinal Collisions

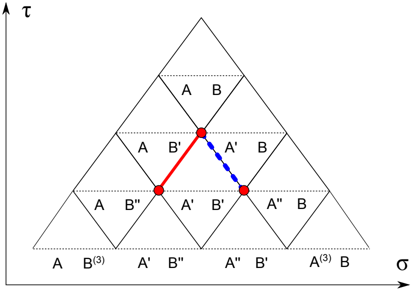
With the string chaos assumption *longitudinal* collisions give

$$f(A,B,t+\Delta t) = \int d\Omega'^2 f(A,B',t)\frac{f(A',B,t)}{\rho(t)}. \qquad (18)$$

Expanding to linear order in time,

$$\frac{\partial f}{\partial t} = \frac{1}{\rho}\int d\Omega'^2\,\Gamma\cdot[f(A,B')f(A',B) - f(A,B)f(A',B')]. \qquad (19)$$

where $1/\Gamma = \Delta t$ is the equilibration time or correlation length.



TRANSVERSE COLLISIONS

- *Transverse* collisions will contribute to the energy density $f(A, B)$ with two terms similar to longitudinal terms.
- So we just expect transverse collisions to add a contribution

$$\Gamma = \frac{1}{\Delta t} \quad \to \quad \Gamma = \frac{1}{\Delta t} + \Gamma_\perp \tag{20}$$

where

$$\Gamma_\perp \propto \rho |A \wedge B \wedge A' \wedge B'|. \tag{21}$$

- This can also be rewritten in terms of the three-velocities and tangent vectors of the interacting segments

$$\Gamma_\perp \propto \rho |(v' - v) \cdot (u' \times u)|. \tag{22}$$

- By denoting the proportionality constant as p (which may also include the inter-commutation probability if desired)

$$\Gamma = \frac{1}{\Delta t} + p\rho |A \wedge B' \wedge A' \wedge B|. \tag{23}$$

H-THEOREM

- Then one can prove H-theorem for strings [VV 2013]

$$\frac{dH}{dt} \leq 0 \leq \frac{dS}{dt}, \tag{24}$$

$$H(t) \equiv \int d\Omega^2 f(A,B) \log(f(A,B) \equiv -S(t) \tag{25}$$

which a *stringy* version of the second law of thermodynamics.

- One can also show [VV 2013] that the distribution relaxes to an equilibrium state with independent statistics of A and B:

$$\lim_{t\to\infty} f(A,B,t) = \frac{1}{\rho} \int d\Omega'^2 f(A,B')f(A',B) \tag{26}$$

or

$$f_{\rm eq}(A,B) = f_A(A) f_B(B) \tag{27}$$

- This allows a significant simplification of the conservation equation leading to a *non-viscous* string fluid dynamics.

Inhomogeneous Limit

- Spatially homogenous transport equation trivially agrees with

$$\nabla_\nu \langle A^\mu B^\nu \rangle = 0 \qquad \nabla_\nu \langle B^\mu A^\nu \rangle = 0 \tag{28}$$

and the next step is to introduce spatial variations.
- For example, one can derive an equation for $f(A, B, x)$ by considering motion of (A, B) segments with velocity $v = \frac{A+B}{2}$.
- This leads to the correct conservation of energy [VV 2013]

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k} \langle v^k \rangle = 0, \tag{29}$$

but does not respect the conservations of A and B fields,

$$\frac{\partial}{\partial t}\langle A^i \rangle + \frac{\partial}{\partial x^k}\langle A^i B^k \rangle = 0 \qquad \frac{\partial}{\partial t}\langle B^i \rangle + \frac{\partial}{\partial x^k}\langle B^i A^k \rangle = 0 \tag{30}$$

- This is a real problem if one wants to use the transport equation to derive *viscous* string fluid equations.

TRANSPORT EQUATION

- The problem is that equations of motion imply that quantities A^i move through space in the direction of B, and vise versa.
- Then the correct difference equation [Schubring, VV 2103]

$$f(A, B, x, t+\Delta t) = \int d\Omega'^2 f(A, B', x - B'\Delta t, t) \frac{f(A', B, x - A'\Delta t, t)}{\rho - \nabla \cdot \langle \mathbf{v} \rangle \Delta t}. \tag{31}$$

and the spatial variations are described with an integral term

$$\boxed{\left(\frac{\partial f}{\partial t}\right)_{\text{spatial}} = -\frac{1}{\rho} \int d\Omega'^2 f(A, B') \overleftrightarrow{\nabla} f(A', B)} \tag{32}$$

where

$$\overleftrightarrow{\nabla} \equiv \overleftarrow{\nabla} \cdot \mathbf{B}' + \mathbf{A}' \cdot \overrightarrow{\nabla} - \frac{1}{\rho} \nabla \cdot \langle \mathbf{v} \rangle \tag{33}$$

- This leads to the transport equation which gives rise to conservation equations in complete agreement with

$$\nabla_\nu \langle A^\mu B^\nu \rangle = 0 \qquad \nabla_\nu \langle B^\mu A^\nu \rangle = 0 \tag{34}$$

Friedmann Universe

- Friedmann metric in conformal coordinates

$$ds^2 = a^2(\tau)(d\tau^2 - dx^2) \tag{35}$$

where $\mathcal{H} \equiv \dot{a}/a$ is the Hubble constant.

- Then the equations of motions

$$B^\mu \partial_\mu A^i = -\mathcal{H}(B^i - (\mathbf{A}\cdot\mathbf{B})A^i) \tag{36}$$

$$A^\mu \partial_\mu B^i = -\mathcal{H}(A^i - (\mathbf{A}\cdot\mathbf{B})B^i). \tag{37}$$

and the change in energy density reduces to,

$$\frac{\dot{\epsilon}}{\epsilon} = -\mathcal{H}(1 + \mathbf{A}\cdot\mathbf{B}). \tag{38}$$

- After somewhat tedious calculations one can show that

$$\boxed{\left(\frac{\partial f}{\partial t}\right)_{\text{gravitational}} = \mathcal{H}\left(\partial_A + \partial_B - (1 + \mathbf{A}\cdot\mathbf{B}) - 4\mathbf{A}\cdot\mathbf{B}\right) f} \tag{39}$$

where

$$\partial_A \equiv (\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{A}) \cdot \frac{\partial}{\partial \mathbf{A}} \tag{40}$$

Fluid Equations

- To verify the conservation equation, we integrate the gravitational terms multiplied by a function $Q(A, B)$:

$$\mathcal{H}\int Q(\partial_A f + \partial_B f)\, d\Omega^2 - \mathcal{H}\langle Q(1 + \mathbf{A}\cdot\mathbf{B})\rangle - 4\mathcal{H}\langle Q(\mathbf{A}\cdot\mathbf{B})\rangle \quad (41)$$

- After integrating by parts and some algebra this becomes,

$$-\mathcal{H}\langle \partial_A Q + \partial_B Q\rangle - \mathcal{H}\langle Q(1 + \mathbf{A}\cdot\mathbf{B})\rangle \quad (42)$$

- Choosing Q to be 1, A^i, B^i we obtain the gravitational correction to the conservation equations,

$$\partial_\nu\langle v^\nu\rangle = -\mathcal{H}\langle 1 + \mathbf{A}\cdot\mathbf{B}\rangle \quad (43)$$

$$\partial_\nu\langle A^i B^\nu\rangle = -2\mathcal{H}\langle v^i\rangle \quad (44)$$

$$\partial_\nu\langle B^i A^\nu\rangle = -2\mathcal{H}\langle v^i\rangle \quad (45)$$

in full agreement with fluid approach [Schubring, VV 2013].

SUMMARY OF RESULTS

Derived a transport equation in the homogeneous limit:

$$\frac{\partial f}{\partial t} = \frac{1}{\rho}\int d\Omega'^2\,\Gamma\cdot[f(A,B')f(A',B) - f(A,B)f(A',B')]$$

$$\text{where } \Gamma = \tfrac{1}{\Delta t} + p\rho|A\wedge B'\wedge A'\wedge B|\,.$$

Proved an H-theorem and derived a local equilibrium distribution:

$$\frac{dH}{dt}\leq 0 \quad \Leftrightarrow \quad f_{\text{eq}}(A,B) = f_A(A)f_B(B)$$

Constructed a correction to the transport eq. due to spatial variations:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{spatial}} = -\frac{1}{\rho}\int d\Omega'^2 f(A,B')(\overleftarrow{\nabla}\cdot\mathbf{B}' + \mathbf{A}'\cdot\overrightarrow{\nabla} - \frac{1}{\rho}\nabla\cdot\langle\mathbf{v}\rangle)f(A',B).$$

Showed that gravitational terms are consistent with conservation eq.:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{gravitational}} = \mathcal{H}\left(\partial_A + \partial_B - (1+\mathbf{A}\cdot\mathbf{B}) - 4\mathbf{A}\cdot\mathbf{B}\right)f$$