Fluid Mechanics of Strings

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Fluid Approach

- Single Nambu-Goto string

$$h_{ab} = g_{\mu\nu} X^{\mu}_{,a} X^{\nu}_{,b}$$

- Energy-momentum tensor over spacetime

$$T^{\mu\nu}(x)\sqrt{-g} = \int d^2\eta \sqrt{-h}\; h^{ab} X^{\mu}_{,a} X^{\nu}_{,b}\; \delta(x - X(\eta))$$

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- Coarse-graining over network

$$\langle T \rangle^{\mu\nu} = \frac{1}{\Delta V} \int T^{\mu\nu} \sqrt{-g}\, d^4x$$

- Fluid equations from conservation equations

$$\nabla_{\mu} T^{\mu\nu} = 0?$$

Singular Currents

- Current confined to worldsheet

$$J^\mu(x) = \int d^2\eta \, \tilde{J}^\mu(\eta) \, \delta(x - X)$$

$$\partial_\mu J^\mu = 0$$

- Flux in coordinate system adapted to worldsheet

$$\frac{\partial \tilde{J}^{\eta^0}}{\partial \eta^0} + \frac{\partial \tilde{J}^{\eta^1}}{\partial \eta^1} = 0$$

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- Back to general coordinates

$$\tilde{J}^\mu = \tilde{J}^{\eta^0} X^\mu_{,0} + \tilde{J}^{\eta^1} X^\mu_{,1}$$

Energy-Momentum Example

- Singular current

$$\tilde{J}^\mu = \tilde{J}^a X^\mu_{,a}$$

$$\partial_a \tilde{J}^a = 0$$

- Energy-momentum tensor

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- Current conservation is equation of motion

$$\partial_a(\sqrt{-h} h^{ab} X^\nu_{,b}) = 0$$

$$\frac{d^2 X^\nu}{d\tau^2} - \frac{d^2 X^\nu}{d\sigma^2} = 0$$

String Current

- Trivial current conservation

$$\frac{d^2 X^\nu}{d\tau d\sigma} - \frac{d^2 X^\nu}{d\sigma d\tau} = 0$$

$$\partial_a(\epsilon^{ab} X^\nu_{,b}) = 0$$

- F-tensor

$$\tilde{F}^{\mu\nu} \equiv \epsilon^{ab} X^\nu_{,b} X^\mu_{,a}$$

- Current conservation

$$F^{\mu\nu}\sqrt{-g} = \int d^2\eta \, \tilde{F}^{\mu\nu} \, \delta(x - X)$$

$$\nabla_\mu F^{\mu\nu} = 0$$

String Flux Example

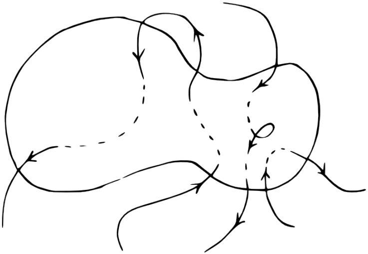
- Time-slice coordinates

$$\tilde{F}^{\mu\nu} = \epsilon^{ab} X^{\mu}_{,a} X^{\nu}_{,b}$$
$$= v^{\mu} u^{\nu} - u^{\mu} v^{\nu}$$

$$v^0 = 1, \quad u^0 = 0$$

- Gauss's Law

$$\partial_\mu F^{\mu 0} = \partial_\mu \langle v^\mu u^0 - u^\mu v^0 \rangle$$
$$= -\partial_\mu \langle u^\mu \rangle$$
$$= 0$$



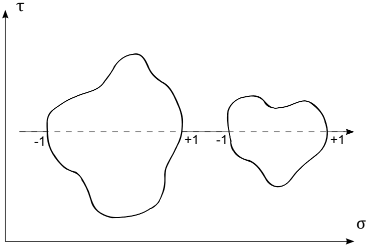
$$\nabla \cdot \langle \mathbf{u} \rangle = 0$$

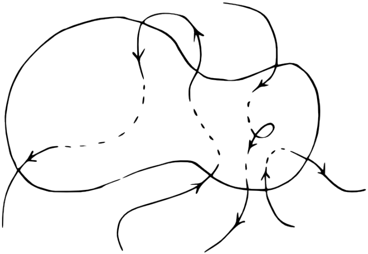
String Flux Example

- Trivial worldsheet current

$$\tilde{j}^\mu = 0\, v^\mu + 1\, u^\mu$$

$$\frac{\partial}{\partial \sigma} 1 = 0$$





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String Flux Example

- If topological monopoles,

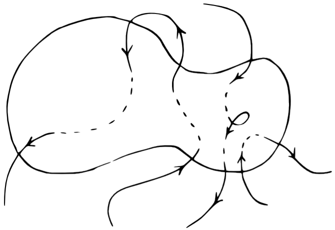
$$\nabla \cdot \langle \mathbf{u} \rangle \neq 0$$

- Local conservation of flux charge

$$\partial_0 \langle \mathbf{u} \rangle = -\nabla \times \langle \mathbf{u} \times \mathbf{v} \rangle$$

- Homogenous Maxwell's equations

$$\nabla_\mu F^{\mu\nu} = 0$$



Light Cone Gauge

- Right and Left Movers ($\mathcal{A}$ and $\mathcal{B}$)

$$\sqrt{-h}h^{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Energy-momentum

$$\tilde{T}^{\mu\nu} = \sqrt{-h}h^{ab}X^{\mu}_{,a}X^{\nu}_{,b} = \mathcal{A}^{\mu}\mathcal{B}^{\nu} + \mathcal{A}^{\nu}\mathcal{B}^{\mu}$$

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- F-tensor

$$\tilde{F}^{\mu\nu} = \epsilon^{ab}X^{\mu}_{,a}X^{\nu}_{,b} = \mathcal{A}^{\mu}\mathcal{B}^{\nu} - \mathcal{A}^{\nu}\mathcal{B}^{\mu}$$

- Full AB-tensor is conserved

Coarse-Graining

- Gauge freedom of $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{A} = \mathcal{A}^0 A^\mu, \quad \mathcal{B} = \mathcal{B}^0 B^\mu$$

- Energy density as measure

$$\begin{aligned} d^2\eta \mathcal{A}^\mu \mathcal{B}^\nu &= d^2\eta\, \mathcal{A}^0 \mathcal{B}^0\, A^\mu B^\nu \\ &= d^2\eta\, \tilde{T}^{00}\, A^\mu B^\nu \end{aligned}$$

- Local equilibrium

$$\begin{aligned} \langle A \otimes B \rangle^{\mu\nu} &= \rho(\overline{A \otimes B})^{\mu\nu} \\ &= \rho \bar{A}^\mu \bar{B}^\nu \end{aligned}$$

- Variance of $\bar{A}$ and $\bar{B}$

Fluid Equations

- Conservation Equations

$$\nabla_\mu(\rho \bar{A}^\mu \bar{B}^\nu) = \nabla_\mu(\rho \bar{B}^\mu \bar{A}^\nu) = 0$$

- Renormalize $\bar{A}$ and $\bar{B}$ as unit vectors, ρ as scalar
- Decoupling of ρ

$$\nabla_\mu(\rho \bar{A}^\mu \bar{B}^\nu) = \nabla_\mu(\rho \bar{A}^\mu)\bar{B}^\nu + \rho \bar{A}^\mu \nabla_\mu \bar{B}^\nu = 0$$

$$\nabla_\mu(\rho \bar{A}^\mu) = 0 \qquad \bar{A}^\mu \nabla_\mu \bar{B}^\nu = 0$$

$$\nabla_\mu(\rho \bar{B}^\mu) = 0 \qquad \bar{B}^\mu \nabla_\mu \bar{A}^\nu = 0$$

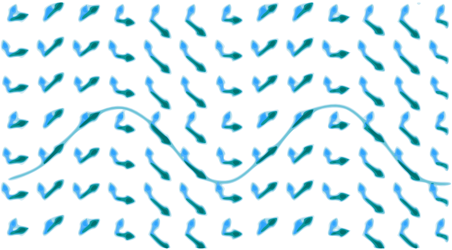
Submanifolds

$$\nabla_\mu(\rho \bar{A}^\mu) = 0 \qquad \bar{A}^\mu \nabla_\mu \bar{B}^\nu = 0$$
$$\nabla_\mu(\rho \bar{B}^\mu) = 0 \qquad \bar{B}^\mu \nabla_\mu \bar{A}^\nu = 0$$

- Frobenius Theorem

$$\bar{A}^\mu \nabla_\mu \bar{B}^\nu - \bar{B}^\mu \nabla_\mu \bar{A}^\nu = 0$$

- Submanifold as worldsheet



Wiggly Strings

- Orthonormal frame

$$v^\mu = \sqrt{v^2}\,\hat{v}^\mu, \qquad u^\mu = \sqrt{-u^2}\,\hat{u}^\mu$$

- Surface energy-momentum tensor

$$\tilde{T}^{\mu\nu} = \sqrt{-h}\,(M\hat{v}^\mu\hat{v}^\nu - T\hat{u}^\mu\hat{v}^\nu)$$

- Finding mass and tension

$$\begin{aligned}\tilde{T}^{\mu\nu} &= \bar{A}^{(\mu}\bar{B}^{\nu)} = v^\mu v^\nu - u^\mu u^\nu \\ &= \sqrt{-h}\,\left(\sqrt{-\frac{v^2}{u^2}}\hat{v}^\mu\hat{v}^\nu - \sqrt{-\frac{u^2}{v^2}}\hat{u}^\mu\hat{u}^\nu\right)\end{aligned} \tag{1}$$

- Equation of state for wiggly string: MT $= 1$ (Vilenkin, 1990)(Carter, 1990)