High Resolution Simulations of Strings in the Abelian Higgs Model

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Outline

Introduction

Abelian Higgs model

Observables

Lattice spacing tests

Physical string width runs

Conclusions

Direct numerical simulation of strings in field theory

- Reliable calculation of string observables requires numerical simulation
- Direct simulation of classical field theory keeps massive radiation
- Challenge: maximising (time)/(string width) while
  - (time) $<$ (box size $N\Delta x$)
  - (grid spacing Δx) $<$ (string width w_{s})
- Challenge in FLRW universe: string width shrinks on comoving lattice
  - Press-Ryden-Spergel ("fat string") algorithm
- Check:
  - Lattice artefacts (vary Δx)
  - Effect of Press-Ryden-Spergel ("fat string") algorithm
- Extrapolation for observables in all simulations requires **scaling**

Direct simulation: standing wave on a smooth string

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Direct simulation: string network from random initial conditions

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Abelian Higgs model: Press-Ryden-Spergel or "fat string" approximation

- Most convenient to solve equations in comoving coordinates ... BUT
- Comoving width of string shrinks as a^{-1}
- Modify (temporal gauge) equations[1]

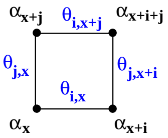
$$\begin{aligned} \ddot{\phi} + 2\frac{\dot{a}}{a}\dot{\phi} - D^2\phi + \lambda a^{2s}(|\phi|^2 - \phi_0^2)\phi &= 0, \\ \partial^\mu \left(\frac{a^{2(1-s)}}{e^2} F_{\mu\nu} \right) - ia^2(\phi^* D_\nu \phi - D_\nu \phi^* \phi) &= 0, \end{aligned}$$

- Physical width of string "fattens" if $s < 1$
- Preserves Gauss's Law (current conservation) but not EM conservation
- Previously: CMB calculated with $s = 0.0, 0.3$
- Now: can simulate with $s = 1$, the physical value.

[1] Press, Ryden, Spergel (1989); Moore, Shellard, Martins (2001); Bevis et al astro-ph/0605018.

String position, winding length

- Lattice: unitary gauge link $\theta^{(u)}_{i,\mathbf{x}} = [\theta_{i,\mathbf{x}} + (\alpha_\mathbf{x} - \alpha_{\mathbf{x}+\mathbf{i}})]_\pi$
- Where $\alpha_\mathbf{x} = [\arg \phi_\mathbf{x}]_\pi$, and $-\pi < [\ldots]_\pi \leq \pi$
- **Winding**: $W_{\langle i,j\rangle,\mathbf{x}} = \left(\theta^{(u)}_{i,\mathbf{x}} + \theta^{(u)}_{j,\mathbf{x}+\mathbf{i}} - \theta^{(u)}_{i,\mathbf{x}_\mathbf{j}} - \theta^{(u)}_{j,\mathbf{x}}\right)/2\pi$
- Strings thread plaquettes with $|W_{\langle i,j\rangle,\mathbf{x}}| \neq 0$
- **Winding length** $L_\mathrm{w} = \Delta x \sum_\Box |W_{\langle i,j\rangle,\mathbf{x}}|$



Scalar densities, Lagrangian length

Lagrangian monomials:

ρ_E	ρ_B	ρ_π	ρ_D	ρ_V
$\frac{1}{V}\sum_{\mathbf{x}} \frac{\mathbf{E}_x^2}{2a^2}$	$\frac{1}{V}\sum_{\mathbf{x}} \frac{\mathbf{B}_x^2}{2a^2}$	$\frac{1}{V}\sum_{\mathbf{x}} \lvert\pi_x\rvert^2$	$\frac{1}{V}\sum_{\mathbf{x}} \lvert\mathbf{D}\phi_x\rvert^2$	$\frac{1}{V}\sum_{\mathbf{x}} a^2 V(\phi)$

- Average Lagrangian density: $\bar{\mathcal{L}} = \rho_E - \rho_B + \rho_\pi - \rho_D - \rho_V$
- Average energy density: $\bar{\rho} = \rho_E + \rho_B + \rho_\pi + \rho_D + \rho_V$
- Average pressure: $\bar{p} = \frac{1}{3}\rho_E + \frac{1}{3}\rho_B + \rho_\pi - \frac{1}{3}\rho_D - \rho_V$

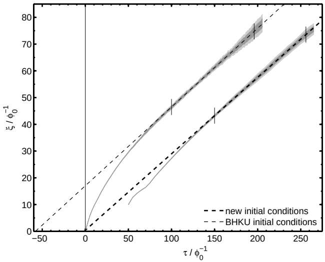
Notes:

- $\mathcal{L}$, ρ, p strongly peaked at strings
- Straight string length L: $\bar{\mathcal{L}} = -\mu L$, where μ = mass per unit length
- Small-amplitude oscillations away from vacuum: $\mathcal{L} \simeq 0$
- **Lagrangian length:** $L_{\mathcal{L}} = -\bar{\mathcal{L}}/\mu$

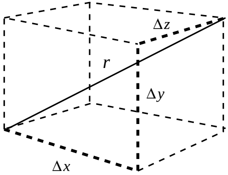
Average string separation ξ

Average string separation: $\xi = \sqrt{V/L}$

- winding length scale ξ_{w}
- Lagrangian length scale $\xi_{\mathcal{L}}$
- In network simulations, $\xi \propto t$ (scaling)
- Scaling from any random initial condition



Note:
"Manhattan" effect: $L_{\mathrm{w}} = |\Delta x| + |\Delta y| + |\Delta z|$
winding length overestimates by factor approx $6/\pi$[2]



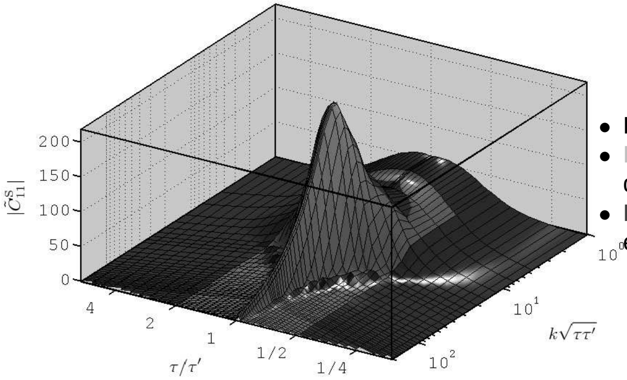
[2] Gaussian random field (Scherrer, Vilenkin 1997)

Energy-momentum correlation functions

- Einstein equations: $(\mathcal{D}h)_{\mu\nu} = (8\pi G)T_{\mu\nu}$
- (statistical) O(3) invariance, EM conservation $\rightarrow$ 6 independent
- Choose to be sources of gauge invariant variables[(3)]
  - Scalar: $\mathcal{S}_\Phi$, $\mathcal{S}_\Psi$
  - Vector: $\mathcal{S}_V^A$ $(A = 1, 2)$
  - Tensor: $\mathcal{S}_T^A$ $(A = 1, 2)$
- **Scaling:** $\langle \mathcal{S}_\alpha(\mathbf{k}, t)\mathcal{S}^*_\beta(\mathbf{k}, t_2)\rangle = \frac{V\phi_0^4}{\sqrt{t_1 t_2}} C_{\alpha\beta}(kt_1, kt_2)$
- **Parity:** only one independent vector and tensor correlator C_V, C_T
- $C_{\alpha\beta}(kt_1, kt_2)$: **unequal** time correlator (UETC)
- $E_{\alpha\beta}(kt) = C_{\alpha\beta}(kt, kt)$: **equal** time correlator (ETC)
- UETCs strongly peaked at ETCs; ETCs peaked at $kt \simeq 2\pi$
- Convention: $C^{AB}_{VV}(kt_1, kt_2) = \sqrt{k^2 t_1 t_2}\delta^{AB} C_V(kt_1, kt_2)$

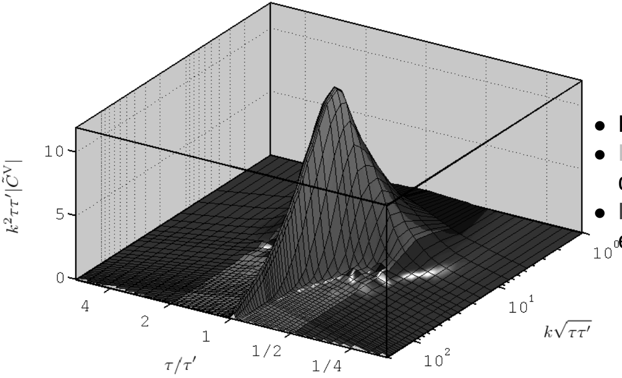
(3) Bardeen; Straumann; Durrer

ΦΦ unequal time correlator (2010)



- Bevis et al 2010
- Light grey: data (1024^3)
- Dark grey: extrapolation

VV unequal time correlator (2010)



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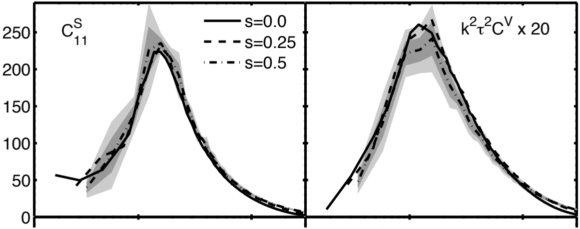
ΦΦ, *VV* equal time correlators (2010)

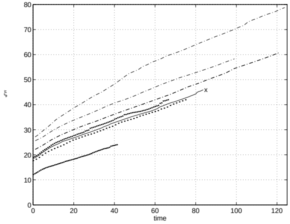


- Bevis et al 2010
- Dark (light) grey: 1σ (2σ) variation
- Physical string width $w_s \propto a(t)^{(1-s)}$

Tests of lattice spacing (in)dependence ($s = 0$)

- Vincent MH Antunes 98:
  - Minkowski space simulations
  - tested Δx independence of $\xi_{\rm w}(t)$

- Now:
  test Δx dependence on (U)ETCs



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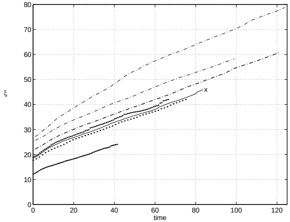

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Units and scales: (Normalisation: $a(t_{\rm end}) = 1$)

Scalar v.e.v.	ϕ_0		1
Scalar coupling	λ		2
Gauge coupling	e		1
Scalar mass	$m_{\rm s}$	$\sqrt{\lambda}a(t)^{s-1}\phi_0$	$\sqrt{2}a(t)^{s-1}$
Vector mass	$m_{\rm v}$	$\sqrt{2}ea(t)^{s-1}\phi_0$	$\sqrt{2}a(t)^{s-1}$
Comoving lattice spacing	Δx		0.5, 0.25, 0.125
String width (lattice units)	$w_{\rm s}$	$m_{\rm v}a(t)^{-s}\Delta x$	0.71, 0.36, 0.18

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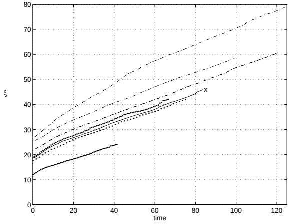
  - $\Delta x = 0.75 \ldots 0.25$



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- Vincent MH Antunes 98:
  - Minkowski space simulations
  - tested Δx independence of $\xi_{\rm w}(t)$
  - $\Delta x = 0.75 \ldots 0.25$
- Bevis et al 2010: $\Delta x = 0.5$, $s = 0$
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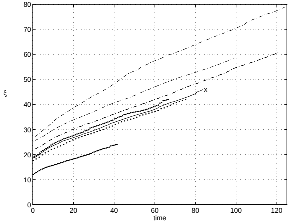

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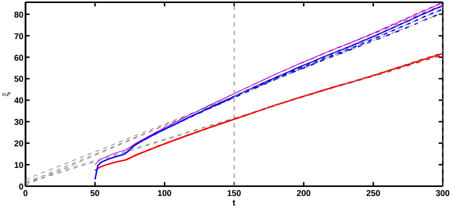
Lattice spacing test strategy

Δx	N	$t = 50$		$50 < t < 70$		$t = 70$		$70 < t < 300$
0.125	4096	ICs	$\rightarrow$	Cool	$\rightarrow$	Decimate	$\rightarrow$	FLRW evolve
						$\downarrow$		
0.25	2048					Decimate	$\rightarrow$	FLRW evolve
						$\downarrow$		
0.5	1024						$\rightarrow$	FLRW evolve

Test in matter era (most important for CMB)

Results: Average string separation ξ



- Solid: $\Delta x = 0.125$
- Dashed: $\Delta x = 0.25$
- Dot-dashed: $\Delta x = 0.5$
- Blue: $\xi_{\mathcal{L}}$
- Red: $\xi_{\rm w}$
- Purple: $\xi_{\rm w}\sqrt{6/\pi}$
- Grey dashed: linear fits

- Linear fits to slope: $150 < t < 300$

Δx	N	$d\xi_{\mathcal{L}}/dt$	$\sqrt{6/\pi}(d\xi_{\rm w}/dt)$
0.125	4096	0.278	0.277
0.25	2048	0.269	0.279
0.5	1024	0.259	0.271

Results: String positions

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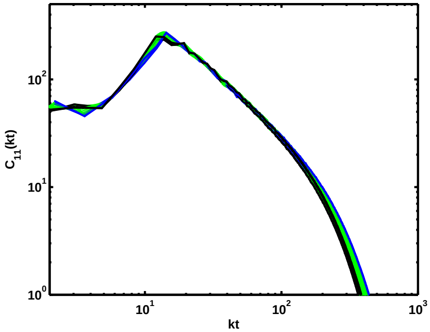
Results: ΦΦ equal time correlator

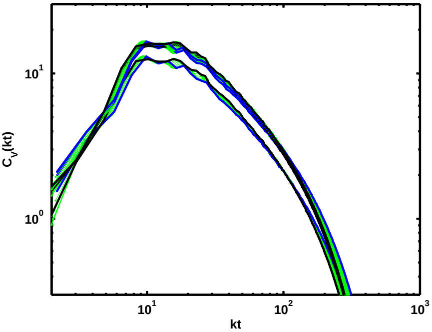
- Black: $t = 150$
- Blue: $t = 180$
- Green: $150 < t < 180$



Results: VV equal time correlator

- Black: $t = 150$
- Blue: $t = 180$
- Green: $150 < t < 180$
- Upper:
  $\Delta x = 0.25$, $\Delta x = 0.125$
- Lower:
  $\Delta x = 0.5$

- Peierls-Nabarro effect
(Strings lose momentum on lattice)
- VV correlator most sensitive
(depends on momentum density T_{0i})



Towards running with physical string width ($s = 1$)

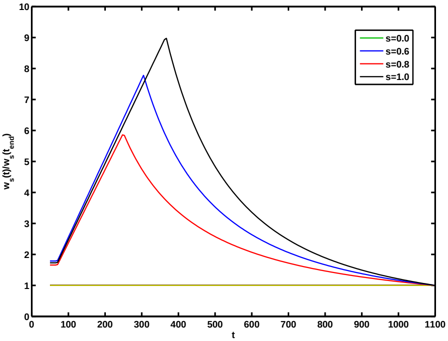
Challenge:

- String width in lattice units
  $w_s(t) = \left(\frac{t_{end}}{t}\right)^2 w_s(t_{end})$
  (matter era)

Solution: [a]

- String width grows initially
- Keep $w_s \Delta x < t/40$

[a] Bevis et al 2007



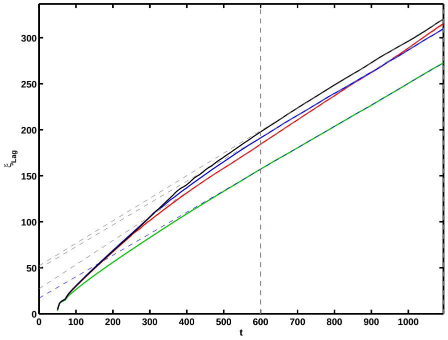
Results: ξ (matter era)

Fits ($600 < t$):

s	$d\xi_{\mathcal{L}}/dt$
1.0	0.246
0.8	0.275
0.6	0.252
0.0	0.245

Mean slopes:

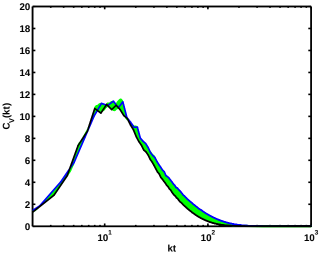
s	$d\xi_{\mathcal{L}}/dt$	N_{runs}
1.0	0.247 ± 0.005	6
0.0	0.234 ± 0.011	7



- Black: $s = 1.0$
- Blue: $s = 0.8$
- Red: $s = 0.6$
- Green: $s = 0.0$

Results: vector ETC (matter era)

- $s = 1$ takes longer to reach scaling
- Scaling form of ETC improved by replacement $t \to \xi(t)$
- Plot ETC as a function of $k\xi$: $\langle S_\alpha(\mathbf{k}, t_1) S^*_\beta(\mathbf{k}, t_2)\rangle = \frac{V\phi_0^4}{\sqrt{\xi_1 \xi_2}} \tilde{C}_{\alpha\beta}(k\xi_1, k\xi_2)$
- Recover correlators from $\xi = \alpha t$: $C(kt_1, kt_2) = \alpha^{-1} C^\xi(\alpha k t_1, \alpha k t_2)$
- Peierls-Nabarro effect small



($600 < t < 700$)

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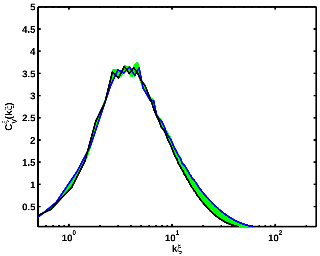

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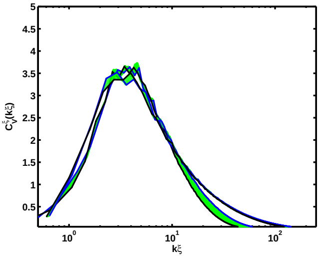

$(600 < t < 700,\ 1000 < t < 1100)$

Conclusions

- Have tested for numerical effects with 4k simulations
- Lattice spacing artefacts $s = 0$:
  - No significant effect on string separation measures ξ_{w}, $\xi_{\mathcal{L}}$
  - No significant effect on UETCs, apart from vector
  - Vector shows a $\sim 20\%$ decrease at $w_{\mathrm{s}}/\Delta x = 1/\sqrt{2}$ (Peierls-Nabarro)
- Simulating at $s = 1$:
  - No significant difference from $s = 0$ for ξ_{w}, $\xi_{\mathcal{L}}$
  - Peierls-Nabarro effect small
  - (U)ETCs: slower approach to scaling at high kt
- $s = 0$ **good, but** $s = 1$ **better!**

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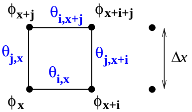
Abelian Higgs model on the lattice (Minkowski space)

- $\phi(\mathbf{x},t) \to \phi_{\mathbf{x}}(t)$, , defined on sites
- Canonical momentum $\pi_{\mathbf{x}}(t) = \dot{\phi}_{\mathbf{x}}(t)$
- $\mathbf{A}(\mathbf{x},t) \to \theta_{i,\mathbf{x}} = -e\Delta x A_{i,\mathbf{x}}(t)$ on links
- Electric field $\epsilon_{i,\mathbf{x}}(t) = \dot{\theta}_{i,\mathbf{x}}$



Discretisation: covariant derivative $\mathbf{D}\phi(\mathbf{x})$, B-field energy density $\frac{1}{2}\mathbf{B}^2$

$|\mathbf{D}\phi(\mathbf{x})|^2 \to \frac{1}{\Delta x^2}\sum_i |e^{-i\theta_{i,\mathbf{x}}}\phi_{\mathbf{x}+\mathbf{i}} - \phi_{\mathbf{x}}|^2$

$\frac{1}{2}\mathbf{B}^2 \to \frac{1}{2\Delta x^4 e^2}\sum_{\langle i,j\rangle}[1-\cos(\theta_{i,\mathbf{x}} + \theta_{j,\mathbf{x}+\mathbf{i}} - \theta_{i,\mathbf{x}+\mathbf{j}} - \theta_{j,\mathbf{x}})]$

Time evolution: Leapfrog. $O(\Delta x^2)$ accurate, conserves (pseudo-)energy.

$$\phi^n_{\mathbf{x}} = \phi^{n-1}_{\mathbf{x}} + \pi^{n-\frac{1}{2}}_{\mathbf{x}}\cdot\Delta t, \qquad \pi^{n+\frac{1}{2}}_{\mathbf{x}} = \pi^{n-\frac{1}{2}}_{\mathbf{x}} + F^n_{\mathbf{x}}\cdot\Delta t,$$

$$\theta^n_{i,\mathbf{x}} = \theta^{n-1}_{i,\mathbf{x}} + \epsilon^{n-\frac{1}{2}}_{i,\mathbf{x}}\cdot\Delta t \qquad \epsilon^{n+\frac{1}{2}}_{i,\mathbf{x}} = \epsilon^{n-\frac{1}{2}}_{i,\mathbf{x}} + G^n_{i,\mathbf{x}}\cdot\Delta t.$$

Preserves discrete version of Gauss's Law $\nabla\cdot E = \rho$.

Abelian Higgs model on the lattice (FLRW spacetime)

Hamiltonian: $\mathcal{H} = \sum_{\mathbf{x}} \left[\frac{1}{2a^2} E_{i,\mathbf{x}}^2 + \frac{1}{2a^2} \mathbf{B}^2 + |\pi_{\mathbf{x}}|^2 + |D\phi|_{i,\mathbf{x}}^2 + a^2 V(\phi_{\mathbf{x}}) \right]$

$$\phi_{\mathbf{x}}^{n} = \phi_{\mathbf{x}}^{n-1} + \pi_{\mathbf{x}}^{n-\frac{1}{2}} \cdot \Delta t, \qquad \pi_{\mathbf{x}}^{n+\frac{1}{2}} = \frac{\pi_{\mathbf{x}}^{n-\frac{1}{2}}(1 - H\Delta t) + F_{\mathbf{x}}^{n} \cdot \Delta t}{1 + H\Delta t},$$

$$\theta_{i,\mathbf{x}}^{n} = \theta_{i,\mathbf{x}}^{n-1} + \epsilon_{i,\mathbf{x}}^{n-\frac{1}{2}} \cdot \Delta t \qquad \epsilon_{i,\mathbf{x}}^{n+\frac{1}{2}} = \frac{\epsilon_{i,\mathbf{x}}^{n-\frac{1}{2}}(1 - (1-s)H\Delta t) + G_{i,\mathbf{x}}^{n} \cdot \Delta t}{1 + (1-s)H\Delta t}.$$

where

$$F_{\mathbf{x}}^{n} = \frac{\partial \mathcal{H}}{\partial \phi_{\mathbf{x}}^{n}}, \quad G_{i,\mathbf{x}}^{n} = \frac{\partial \mathcal{H}}{\partial \theta_{i,\mathbf{x}}^{n}} \qquad (1)$$